Triples Fixed Point Results in Generalized Metric Spaces
Under Nonlinear Type Contractions Depended on
Another Function

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ABSTRACT
In 2006, Mustafa and Sims [18-19] introduced an improved version of the generalized metric space structure which they called G-metric spaces and in 2011; Berinde and Borcut [11] introduced the concept of triple fixed point. The intent of this paper is to establish some tripled fixed point theorems for mappings having mixed monotone property under nonlinear type contractions depended on another function in the framework of a G-metric space \(X\) enclosed with partial order. The presented results generalize, improve and extend corresponding results of H. Aydi et al. [13] (Triples Fixed Point Results in Generalized Metric Spaces" Journal of Applied Mathematics Volume 2012, Article ID 314279, 10 pages, doi:10.1155/2012/314279). Moreover, some examples are provided to illustrate the usability of the obtained results.

Keywords and Phrases: Triples fixed point; nonlinear contractions; partially ordered sets; G-metric spaces; mixed monotone; ICS mapping.

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1. INTRODUCTION
Fixed point theory has fascinated many researchers since 1922 with the celebrated Banach’s fixed point theorem. There exists a vast literature on the topic and is a very active field of research at present. Theorems concerning the existence and properties of fixed points are known as fixed point theorems. Such theorems are very important tool for proving the existence and eventually the uniqueness of the solutions to various mathematical models (integral and partial differential equations, variational inequalities).

the concept of triple fixed point and established some triple fixed point theorems in partially ordered metric spaces. A tripled fixed point is a generalization of the well-known concept of coupled fixed point. The study of tripled fixed point is a very interesting research area in fixed point theory.

The notion of D-metric space is a generalization of usual metric spaces and it is introduced by Dhaage [14-17]. Recently, Mustafa and Sims [18-19] have shown that most of the results concerning Dhaage’s D-metric spaces are invalid. In [18-19], they introduced an improved version of the generalized metric space structure which they called G-metric spaces. For more results on G-metric spaces, one can refer to the papers [24-30, 33-35]. Hassen et al. [13] established some tripled fixed point results in G-Metric Spaces.

The purpose of this paper is three fold which can be described as follows.

1. We give some example which shows the weakness of Theorem 16 and Theorem 17 (Theorem 2.1 and 2.4 of Hassen et al. [13]).

2. We establish some tripled fixed point theorems for continuous mappings having mixed monotone property under nonlinear type contractions depended on another function T: X → X (T is an ICS) in the frame work of a G-metric space X enclosed with partial order. Also these theorems, are still valid for F, not necessarily continuous, assuming (X, G, ≤) is regular. We prove the uniqueness of tripled fixed point for such mappings in this setup. Our results are extensions of the main results of Hassen et al. [13].

3. We present some examples to illustrate the effectiveness of our results. Also, we give a simple example which shows that if T is not an ICS mapping then the conclusion of main results fail.

2. DEFINITIONS AND PREMILINARIES

Throughout this paper, we denote ℜ₊ the set of all positive real numbers and ℕ the set of all natural numbers. The triple (X, G, ≤) is called a partially ordered G-metric space if (X, ≤) is a partially ordered set and (X, G) is a G-metric space. Further, if (X, G) is complete metric space, and then the triple (X, G, ≤) is called a partially ordered complete G-metric space. We assume that X ≠ ∅ and

\[ X^k = X \times X \times \ldots \times X \quad (k\text{-times}) \]  

In what follows, we collect some relevant definitions, fundamental results, examples for our further use.

**Definition 1** (2006, Mustafa and Sims [19]) Let X be a nonempty set, and let G: X × X × X → ℜ₊ be a function satisfying the following properties:

(G1) \( G(x, y, z) = 0 \) if \( x = y = z \),
(G2) \( 0 < G(x, x, y) \) for all \( x, y \) ∈ X, with \( x ≠ y \),
(G3) \( G(x, y, z) ≤ G(x, y, y) + G(y, z, z) \) for all \( x, y, z \) ∈ X, with \( x ≠ y \),
(G4) \( G(x, y, z) = G(x, z, y) = G(y, z, x) = \ldots \) (Symmetry in all three variables),
(G5) \( G(x, y, z) ≤ G(x, a, a) + G(a, y, z), \forall x, y, z, a ∈ X \), (Rectangle inequality).

Then the function G is called generalized metric or more specially, G-metric on X, and the pair (X, G) is called a G-metric space. Every G-metric on X will define a metric \( d_G \) on X by

\[ d_G(x, y) = G(x, y, y) \]  

**Example 2** (Hassen et al. [13]) Let (X, d) be a metric space. The function G: X × X × X → ℜ₊ defined by

\[ G(x, y, z) = \max\{d(x, y), d(y, z), d(z, x)\} \]  

or

\[ G(x, y, z) = d(x, y) + d(y, z) + d(z, x) \]  

for all \( x, y, z \) ∈ X, is a G-metric on X.

**Definition 3** (see [19]) Let (X, G) be a G-metric space and let \( \{x_n\} \) be a sequence of points of X, a point x ∈ X is said to be the limit of the sequence \( \{x_n\} \) if \( \lim_{n \to \infty} G(x_n, x, x) = 0 \), and one say that the sequence \( \{x_n\} \) is G-convergent to x. Thus, that if \( x_n \to x \) in a G-metric space (X, G), then for any \( \varepsilon > 0 \), there exists N ∈ ℕ such that \( G(x_n, x, x) < \varepsilon \), for all n, m ≥ N.

**Proposition 4** (see [19]) Let (X, G) be a G-metric space. The following are equivalent:

1. \( \{x_n\} \) is G-convergent to x,
2. \( G(x_n, x, x) \to 0 \) as \( n \to +\infty \),
3. \( G(x_n, x, x) \to 0 \) as \( n \to +\infty \),
4. \( G(x_n, x, x) \to 0 \) as \( n, m \to +\infty \).

**Definition 5** (see [19]) Let (X, G) be a G-metric space. A sequence \( \{x_n\} \) is called G-Cauchy if given \( \varepsilon > 0 \), there is N ∈ ℕ such that \( G(x_n, x_m, x_l) < \varepsilon \), for all n, m, l ≥ N, that is, if \( G(x_n, x_m, x_l) \to 0 \) as \( n, m, l \to +\infty \).

**Proposition 6** (see [19]) Let (X, G) be a G-metric space. The following are equivalent:

1. The sequence \( \{x_n\} \) is G-Cauchy.
2. For every \( \varepsilon > 0 \), there exists N ∈ ℕ such that \( G(x_n, x_m, x_l) < \varepsilon \) for all n, m, l ≥ N.

**Definition 7** (see [19]) Let (X, G) and (X’, G’) are two G-metric spaces, and let f: (X, G) → (X’, G’) be a function. Then f is said to be G-continuous at a point a ∈ X if and only if given \( \varepsilon > 0 \), there exists \( \delta > 0 \) such that \( x, y \in X; G(a, x, y) < \delta \) implies \( G'(f(a), f(x), f(y)) < \varepsilon \). A function f is G-continuous on X if and only if it is G-continuous at all a ∈ X.

**Proposition 8** (see [19]) Let (X, G) and (X’, G’) be two G-metric spaces. Then a function f: X → X’ is G-continuous at a point x ∈ X if and only if it is G-sequentially continuous at x; that is, whenever \( \{x_n\} \) is G-convergent to x, we have \( \{f(x_n)\} \) is G’-convergent to f(x).
Proposition 9 (see [19]) Let (X, G) be a G-metric space. Then the function G(x, y, z) is jointly continuous in all three of its variables. ■

Definition 10 (see [19]) A G-metric space (X, G) is said to be G-complete (or complete G-metric space) if every G-Cauchy sequence is G-convergent in (X, G).

Definition 11 (see [19]) A G-metric space (X, G) is called symmetric G-metric space if G(y, x, x) = G(x, y, y) for all x, y ∈ X.

Proposition 9 Let (X, G) be a partially ordered set and suppose there is a G-metric G on X such that GCauchy sequence is G-convergent in (X, G).

Theorem 16 (Hassen et al. [13]) Let (X, ≤) be partially ordered set and suppose there is a G-metric G on X such that (X, G) is a G-complete. Suppose also F: X³ → X be a mapping having the mixed monotone property on X. Assume that there exists ϕ ∈ Φ such that for x, y, z, a, b, c, u, v, w ∈ X, with x ≥ a ≥ u, y ≤ b ≤ v, and z ≥ c ≥ w, one has (7). If there exist x₀, y₀, z₀ ∈ X such that

x₀ ≤ F(x₀, y₀, z₀), y₀ ≥ F(y₀, x₀, y₀),

z₀ ≤ F(z₀, y₀, x₀). (8)

Then F has a tripled fixed point in X, that is, there exist x, y, z ∈ X such that

F(x, y, z) = x, F(y, x, y) = y, F(z, y, x) = z. ■

Theorem 17 (see [13]) Let (X, ≤) be partially ordered set and suppose there is a G-metric G on X such that (X, G) is a G-complete. Suppose also F: X³ → X be a mapping having the mixed monotone property on X. Assume that there exists ϕ ∈ Φ such that for x, y, z, a, b, c, u, v, w ∈ X, with x ≥ a ≥ u, y ≤ b ≤ v, and z ≥ c ≥ w, one has (7). If there exist x₀, y₀, z₀ ∈ X such that

x₀ ≤ F(x₀, y₀, z₀), y₀ ≥ F(y₀, x₀, y₀),

z₀ ≤ F(z₀, y₀, x₀).

Assume also that X has the following properties:

a) if a non-decreasing sequence xₙ → x in X, then xₙ ≤ x, ∀ n.
b) if a non-increasing sequence yₙ → y in X, then yₙ ≥ y, ∀ n.

Then F has a tripled fixed point in X, that is, there exist x, y, z ∈ X such that

F(x, y, z) = x, F(y, x, y) = y, F(z, y, x) = z. ■

3. MAIN RESULTS

We start this section with some examples which shows the weakness of Theorem 16 and 17 (Theorem 2.1 and 2.4 in [13]).

Example 18(a) Take X = [1/7, 64] endowed with the complete G-metric:

G(x, y, z) = |x − y| + |y − z| + |z − x|, ∀ x, y, z ∈ X

and F: X³ → X be defined by

F(x, y, z) = 8 \left( \frac{x}{y} \right), \ ∀ x, y, z ∈ X.

The mapping F is continuous and has the mixed monotone property. Also, there exist x₀ = 1 = z₀ and y₀ = 64 such that

F(x₀, y₀, z₀) = F(1, 64, 1) = 8 \left( \frac{1}{64} \right) = 4 > 1 = x₀,

F(y₀, x₀, y₀) = F(64, 1, 64) = 8 \left( \frac{1}{64} \right) = 16 < 64 = y₀,

F(z₀, y₀, x₀) = F(1, 64, 1) = 8 \left( \frac{1}{64} \right) = 4 > 1 = z₀.

and then, the condition (8) holds. Taking x = w = a = c = z = 1, y = b = v = 64, u = 1/2

G(F(x, y, z), F(a, b, c), F(u, v, w))
Notice, however, that

Now, motivated by the work in [31-32], we give the point of F.

It is clear that here is no \( \phi \in \Phi \) (no \( k \in [0,1) \)) for which the inequality (2.1) (inequalities (2.14) and (2.16)) of Theorem 2.1 (Corollary 2.2 and 2.3) holds of [13]. Notice, however, that (8,8,8) is the unique triped fixed point of F.

**Example 18(b)** Take \( X = [1,64] \) endowed with the complete G-metric:

\[
G(x,y,z) = |x - y| + |y - z| + |z - x|, \forall x,y,z \in X
\]

and \( F:X^3 \rightarrow X \) defined by

\[
F(x,y,z) = 8 \frac{1}{4} \sqrt[3]{5}, \forall x,y,z \in X.
\]

The mapping F is continuous and has the mixed monotone property. Moreover, taking \( x_0 = y_0 = z_0 = 8 \), the condition (8) holds. Taking \( u = 1, a = x = 2, v = 2, b = z = c = w = 1 \),

\[
G(F(x,y,z), F(a,b,c), F(u,v,w)) = 8 \left( \left( \frac{2}{3} \right)^2 - \left( \frac{1}{3} \right)^2 \right) + \left( \frac{1}{2} \right)^2 - (2)^2) \right) \approx 7.458
\]

and

\[
\max(G(x,a,u), G(y,b,v), G(z,c,w)) = 2.
\]

It is clear that here is no \( \phi \in \Phi \) (no \( k \in [0,1) \)) for which the inequality (2.1) (inequalities (2.14) and (2.16)) of Theorem 2.1 (Corollary 2.2 and 2.3) holds of [13]. Notice, however, that (8,8,8) is the unique triped fixed point of F.

Now, motivated by the work in [31-32], we give the following definition.

**Definition 19** Let \( (X,G) \) be a G-metric space. A mapping \( T:X \rightarrow X \) is said to be an ICS mapping if \( T \) is injective, continuous, and has the property: for every sequence \( \{x_n\} \) in \( X \), if \( \{Tx_n\} \) is convergent then, \( \{x_n\} \) is also convergent.

Before starting to introduce our main results, let us consider the set of functions.

\[
\Phi = \{ \phi: \mathbb{R}^+ \rightarrow \mathbb{R}^+ | \phi \text{ is nondecreasing, } \phi(t) < t \text{ and } \lim_{t \rightarrow 0^+} \phi(t) < t, \forall \ t > 0 \} \quad (9)
\]

Note that \( \phi(t) < t \) and \( \lim_{t \rightarrow 0^+} \phi(t) < t \) imply \( \lim_{k \rightarrow 0^+} \phi^k(t) = 0 \) for each \( t > 0 \), where \( \phi^k \) denotes the \( k \)-times composition of \( \phi \) with itself.

Our first main result is given by the following:

**Theorem 20** Let \( (X,\leq) \) be partially ordered set and suppose there is a G-metric \( G \) on \( X \) such that \( (X,G) \) is a G-complete. Suppose also that \( T:X \rightarrow X \) is an ICS mapping and \( F:X^3 \rightarrow X \) be a continuous mapping having the mixed monotone property on \( X \). Assume that there exists \( \phi \in \Phi \) such that for \( x,y,z,a,b,c,u,v,w \in X \) with \( x \geq a \geq u \), \( y \leq b \leq v \), and \( z \geq c \geq w \), one has

\[
G(F(x,y,z), F(a,b,c), F(u,v,w)) \leq \phi\left( \max\left( G(Tx, Ta, Tu), G(Ty, Tb, Tv), G(Tz, Tc, Tw) \right) \right)
\]

\[
G(Tx, Ty, Tz) \leq kG(x,y,z)
\]

where \( k \in (0,1) \).

Then \( x_0, y_0, z_0 \in X \) be as in (8). Then \( F \) has a triped fixed point in \( X \).

**Proof:** Suppose \( x_0, y_0, z_0 \in X \) are such that

\[
x_0 \leq F(x_0, y_0, z_0), \quad y_0 \geq F(y_0, x_0, y_0), \quad z_0 \leq F(z_0, y_0, x_0)
\]

Define

\[
x_1 = F(x_0, y_0, z_0), \quad y_1 = F(y_0, x_0, y_0), \quad z_1 = F(z_0, y_0, x_0)
\]

Then \( x_0 \leq x_1 \leq x_2 \). Again, define

\[
x_2 = F(x_1, y_1, z_1), \quad y_2 = F(y_1, x_1, y_1), \quad z_2 = F(z_1, y_1, z_1)
\]

Since \( F \) has the mixed monotone property on \( X \), we have

\[
x_0 \leq x_1 \leq x_2 \leq y_1 \leq y_0 \leq z_0 \leq z_1 \leq z_2.
\]

Repeating this process, we can construct the sequences \( \{x_n\}, \{y_n\}, \) and \( \{z_n\} \) in \( X \) such that

\[
x_{n+1} = F(x_n, y_n, z_n), \quad y_{n+1} = F(y_n, x_n, y_n), \quad z_{n+1} = F(z_n, y_n, x_n), \quad \forall \ n \geq 0.
\]

We claim that:

\[
x_n \leq x_{n+1}, \quad y_{n+1} \leq y_n, \quad z_n \leq z_{n+1}. \quad (12)
\]

Indeed, we will use the mathematical induction to prove (12). The inequalities in (12) hold for \( n = 1,2 \) because, we have \( x_0 \leq x_1 \leq x_2 \leq y_1 \leq y_0 \) and \( z_0 \leq z_1 \leq z_2 \). Now, suppose that the inequalities in (12) hold for \( n = m \). In that case,

\[
x_m \leq x_{m+1}, \quad y_{m+1} \leq y_m, \quad z_m \leq z_{m+1}. \quad (13)
\]
If we consider (11) and mixed monotone property of F together with (13), we have

\[
\begin{align*}
x_{m+1} &= F(x_m,y_m,z_m) \\
&\leq F(x_{m+1},y_{m+1},z_{m+1}) \\
&\leq F(x_m,y_m,z_m) = x_{m+2},
\end{align*}
\]

\[
\begin{align*}
y_{m+1} &= F(y_m,x_m,y_m) \\
&\geq F(y_{m+1},x_{m+1},y_{m+1}) \\
&\geq F(y_m,x_m,y_m) = y_{m+2},
\end{align*}
\]

\[
\begin{align*}
z_{m+1} &= F(z_m,x_m,y_m) \\
&\leq F(z_{m+1},y_{m+1},x_{m+1}) \\
&\leq F(z_m,x_m,y_m) = z_{m+2}.
\end{align*}
\]

Thus, (12) is satisfied for all \(n \geq 1\). If for some positive integer \(n\), we have \((x_{n+1},y_{n+1},z_{n+1}) = (x_n,y_n,z_n)\), then \(x_n = F(x_n,y_n,z_n)\), \(y_n = F(y_n,x_n,y_n)\), and \(z_n = F(z_n,x_n,z_n)\), that is, \((x_n,y_n,z_n)\) is a tripled fixed point of \(F\). Thus, we will assume that \((x_{n+1},y_{n+1},z_{n+1}) \neq (x_n,y_n,z_n)\) for all \(n \in \mathbb{N}\). Since \(T\) is injective, for any \(n \in \mathbb{N}\),

\[
0 < \max\{G(Tx_{n+1},Tx_n),G(Ty_{n+1},Ty_n),G(Tz_{n+1},Tz_n)\} \leq \phi(\max\{G(Tx_n,Tx_{n-1}),G(Ty_n,Ty_{n-1}),G(Tz_n,Tz_{n-1})\})
\]

Due to (10) and (11), for any \(n \in \mathbb{N}\), we have

\[
\begin{align*}
G(Tx_{n+1},Tx_n) &= G(TF(x_n,y_n,z_n),TF(x_{n-1},y_{n-1},z_{n-1}),TF(x_{n-1},y_{n-1},z_{n-1})) \\
&\leq \phi(\max\{G(Tx_n,Tx_{n-1}),G(Ty_n,Ty_{n-1}),G(Tz_n,Tz_{n-1})\})
\end{align*}
\]

\[
\begin{align*}
G(Ty_{n+1},Ty_n) &= G(TF(y_n,x_n,y_n),TF(y_{n-1},x_{n-1},y_{n-1}),TF(y_{n-1},x_{n-1},y_{n-1})) \\
&\leq \phi(\max\{G(Ty_n,Ty_{n-1}),G(Ty_n,Ty_{n-1}),G(Ty_n,Ty_{n-1})\})
\end{align*}
\]

\[
\begin{align*}
G(Tz_{n+1},Tz_n) &= G(TF(z_n,y_n,x_n),TF(z_{n-1},y_{n-1},x_{n-1}),TF(z_{n-1},y_{n-1},x_{n-1})) \\
&\leq \phi(\max\{G(Tz_n,Tz_{n-1}),G(Tz_n,Tz_{n-1}),G(Tz_n,Tz_{n-1})\})
\end{align*}
\]

Having in mind that \(\phi(t) < t\) for all \(t > 0\), so from (16), we obtain that

\[
0 < \max\{G(Tx_{n+1},Tx_n),G(Ty_{n+1},Ty_n),G(Tz_{n+1},Tz_n)\} \leq \phi(\max\{G(Tx_n,Tx_{n-1}),G(Ty_n,Ty_{n-1}),G(Tz_n,Tz_{n-1})\})
\]

or

\[
0 < \max\{G(Tx_{n+1},Tx_n),G(Ty_{n+1},Ty_n),G(Tz_{n+1},Tz_n)\} \leq \phi(\max\{G(Tx_n,Tx_{n-1}),G(Ty_n,Ty_{n-1}),G(Tz_n,Tz_{n-1})\})
\]

This means there exists an \(\varepsilon > 0\) for which we can find subsequences \((T_{n_k})\) of \((T_{x_n})\), \((T_{y_m})\) of \((T_{y_n})\) and \((T_{z_m})\) of \((T_{z_n})\) with \(n_k > m_k > k\) such that

\[
\max\{G(Tx_{n_k},Tx_{m_k}),G(Ty_{n_k},Ty_{m_k}),G(Tz_{n_k},Tz_{m_k})\} \geq \varepsilon.
\]
Further, corresponding to $m_k$, we can choose $n_k$ in such a way that it is the smallest integer with $n_k > m_k \geq k$ satisfying (23). Then

$$\max\{G(Tx_{n_k-1}, Tx_{m_k}, Tx_{m_k}),$$

$$G(Ty_{n_k-1}, Ty_{m_k}, Ty_{m_k}),$$

$$G(Tz_{n_k-1}, Tz_{m_k}, Tz_{m_k})\} < \varepsilon.$$  \hfill (24)

By rectangle inequality and (24), we have

$$G(Tx_{m_k}, Tx_{m_k}, Tx_{n_k}) \leq G(Tx_{m_k}, Tx_{m_k}, Tx_{n_k-1})$$

$$+ G(Tx_{n_k-1}, Tx_{n_k-1}, Tx_{m_k})$$

$$\leq \varepsilon + G(Tx_{n_k-1}, Tx_{n_k-1}, Tx_{m_k})$$  \hfill (25)

Letting $k \to +\infty$ in (25) and using (21), we obtain

$$\lim_{k \to +\infty} G(Tx_{m_k}, Tx_{m_k}, Tx_{n_k}) \leq \lim_{k \to +\infty} G(Tx_{m_k}, Tx_{m_k}, Tx_{n_k-1}) \leq \varepsilon.$$  \hfill (26)

Similarly,

$$\lim_{k \to +\infty} G(Tx_{m_k}, Tx_{m_k}, Tx_{n_k}) \leq \lim_{k \to +\infty} G(Tx_{m_k}, Tx_{m_k}, Tx_{n_k-1}) \leq \varepsilon.$$  \hfill (27)

Again, by (24), we have

$$\varepsilon \leq G(Tx_{m_k}, Tx_{m_k}, Tx_{n_k})$$

$$\leq G(Tx_{m_k}, Tx_{m_k}, Tx_{m_k-1}) + G(Tx_{m_k-1}, Tx_{m_k-1}, Tx_{n_k})$$

$$\leq G(Tx_{m_k}, Tx_{m_k}, Tx_{m_k-1})$$

$$+ G(Tx_{m_k-1}, Tx_{m_k-1}, Tx_{n_k-1}) + G(Tx_{n_k-1}, Tx_{n_k-1}, Tx_{m_k-1})$$

$$\leq G(Tx_{m_k}, Tx_{m_k}, Tx_{m_k-1})$$

$$+ G(Tx_{m_k-1}, Tx_{m_k-1}, Tx_{n_k-1}) + G(Tx_{n_k-1}, Tx_{n_k-1}, Tx_{m_k-1})$$

$$\leq G(Tx_{m_k}, Tx_{m_k}, Tx_{m_k-1})$$

$$+ G(Tx_{m_k-1}, Tx_{m_k-1}, Tx_{n_k-1}) + G(Tx_{n_k-1}, Tx_{n_k-1}, Tx_{m_k-1})$$

$$+ \varepsilon + G(Tx_{n_k-1}, Tx_{n_k-1}, Tx_{m_k})$$  \hfill (28)

Letting $k \to +\infty$ in (28) and using (21), we get

$$\varepsilon \leq \lim_{k \to +\infty} G(Tx_{m_k}, Tx_{m_k}, Tx_{n_k})$$

$$\leq \lim_{k \to +\infty} G(Tx_{m_k-1}, Tx_{m_k-1}, Tx_{n_k-1}) \leq \varepsilon.$$  \hfill (29)

Similarly, we have

$$\varepsilon \leq \lim_{k \to +\infty} G(Ty_{m_k}, Ty_{m_k}, Ty_{n_k})$$

$$\leq \lim_{k \to +\infty} G(Ty_{m_k-1}, Ty_{m_k-1}, Ty_{n_k-1}) \leq \varepsilon,$$

$$\varepsilon \leq \lim_{k \to +\infty} G(Tz_{m_k}, Tz_{m_k}, Tz_{n_k})$$

$$\leq \lim_{k \to +\infty} G(Tz_{m_k-1}, Tz_{m_k-1}, Tz_{n_k-1}) \leq \varepsilon.$$  \hfill (30)

Set $r_k = \max\{G(Tx_{n_k}, Tx_{m_k}, Tx_{m_k}),$

$$G(Ty_{n_k}, Ty_{m_k}, Ty_{m_k}),$

$$G(Tz_{n_k}, Tz_{m_k}, Tz_{m_k})\}$. Using (23) and (29)-(30), we have

$$\varepsilon = \lim_{k \to +\infty} r_k.$$  \hfill (31)

Now, using inequality (10), we obtain

$$G(Tx_{n_k}, Tx_{m_k}, Tx_{m_k}) = G(TF(x_{n_k-1}, y_{m_k-1}, z_{n_k-1}),$$

$$TF(x_{m_k-1}, y_{m_k-1}, z_{m_k-1}),$$

$$TF(x_{m_k-1}, y_{m_k-1}, z_{m_k-1}))$$

$$\leq \phi(\max\{G(Tx_{n_k-1}, Tx_{m_k-1}, Tx_{m_k-1}),$$

$$G(Ty_{n_k-1}, Ty_{m_k-1}, Ty_{m_k-1}),$$

$$G(Tz_{n_k-1}, Tz_{m_k-1}, Tz_{m_k-1})\})$$

$$G(Ty_{n_k}, Ty_{m_k}, Ty_{m_k}) = G(TF(x_{n_k-1}, y_{m_k-1}, y_{m_k-1}),$$

$$TF(y_{m_k-1}, y_{m_k-1}, y_{m_k-1}),$$

$$TF(y_{m_k-1}, y_{m_k-1}, y_{m_k-1}))$$

$$\leq \phi(\max\{G(Ty_{n_k-1}, Ty_{m_k-1}, Ty_{m_k-1}),$$

$$G(Tx_{n_k-1}, Ty_{m_k-1}, Ty_{m_k-1}),$$

$$G(Tz_{n_k-1}, Ty_{m_k-1}, Ty_{m_k-1})\})$$

$$G(Tz_{n_k}, Tz_{m_k}, Tz_{m_k}) = G(TF(z_{n_k-1}, z_{n_k-1}, y_{m_k-1}),$$

$$TF(z_{m_k-1}, z_{m_k-1}, y_{m_k-1}),$$

$$TF(z_{m_k-1}, z_{m_k-1}, y_{m_k-1}))$$

$$\leq \phi(\max\{G(Tz_{n_k-1}, Tz_{m_k-1}, Tz_{m_k-1}),$$

$$G(Tx_{n_k-1}, Tz_{m_k-1}, Tz_{m_k-1}),$$

$$G(Ty_{n_k-1}, Tz_{m_k-1}, Tz_{m_k-1})\})$$  \hfill (32)

From (32), we deduce that

$$\max\{G(Tx_{n_k}, Tx_{m_k}, Tx_{m_k}),$$

$$G(Ty_{n_k}, Ty_{m_k}, Ty_{m_k}),$$

$$G(Tz_{n_k}, Tz_{m_k}, Tz_{m_k})\}$$

$$\leq \phi(\max\{G(Tx_{n_k-1}, Tx_{m_k-1}, Tx_{m_k-1}),$$

$$G(Ty_{n_k-1}, Ty_{m_k-1}, Ty_{m_k-1}),$$

$$G(Tz_{n_k-1}, Tz_{m_k-1}, Tz_{m_k-1})\})$$

That is,

$$r_k \leq \phi(r_{k-1}).$$  \hfill (33)

Letting $k \to +\infty$ in (33) and having in mind (31), we get that

$$0 < \varepsilon = \lim_{k \to +\infty} r_k$$

$$\leq \lim_{k \to +\infty} \phi(r_{k-1})$$

$$= \lim_{k \to +\infty} \varepsilon + \phi(r_{k-1}) < \varepsilon,$$

which is a contradiction. Thus, $\{Tx_n\}, \{Ty_n\}$, and $\{TZ_n\}$ are G-Cauchy sequences in $(X, G)$. Since $(X, G)$ is a G-
complete, \(\{Tx_n\}, \{Ty_n\}\) and \(\{Tz_n\}\) are convergent sequences.

Since \(T\) is an ICS mapping, there exist \(x, y, z \in X\) such that \([x_n], [y_n]\), and \([z_n]\) converge to \(x, y, \) and \(z\), respectively; that is,

\[
\lim_{n \to +\infty} x_n = x, \\
\lim_{n \to +\infty} y_n = y, \\
\lim_{n \to +\infty} z_n = z.
\]

(34)

Finally, we show that \((x, y, z) \in X^3\) is a tripled fixed point of \(F\). Since \(F\) is continuous and \(x_n, y_n, z_n\) converge to \(x, y, \) and \(z\), respectively; that is,

\[
x_{n+1} = F(x_n, y_n, z_n) \to F(x, y, z),
\]

\[
y_{n+1} = F(y_n, x_n, y_n) \to F(y, x, y),
\]

\[
z_{n+1} = F(z_n, z_n, y_n) \to F(z, z, y).
\]

By the uniqueness of limit, we get that \(x = F(x, y, z),\)

\(y = F(y, x, y),\) and \(z = F(z, z, y)\). So \((x, y, z)\) is a tripled fixed point of \(F\). This completes the proof. ■

**Corollary 21** Let \((X, \leq)\) be partially ordered set and suppose there is a G-metric \(G\) on \(X\) such that \((X, G)\) is a G-complete. Suppose also that \(T: X \to X\) is an ICS mapping and \(F: X^3 \to X\) be a continuous mapping having the mixed monotone property on \(X\). Assume that there exists \(\phi \in \Phi\), such that \(\phi(x, y, z, a, b, c, u, v, w) \in X\), with \(x \geq a \geq u, y \leq b \leq v, \) and \(z \geq c \geq w,\) one has

\[
G(TF(x, y, z), TF(a, b, c), TF(u, v, w) \leq \phi \left( G(Tx, Ta, Tu), G(Ty, Tb, Tv), G(Tz, Tc, Tw) \right)
\]

(35)

If there exist \(x_0, y_0, z_0 \in X\) be as in (8). Then \(F\) has a tripled fixed point in \(X\).

**Proof:** It suffices to remark that

\[
\frac{G(Tx, Ta, Tu) + G(Ty, Tb, Tv) + G(Tz, Tc, Tw)}{3}
\]

Then, we apply Theorem 20 because that \(\phi\) is non-decreasing. ■

For each \(k \in [0,1]\), setting \(\phi(t) = kt\) in Theorem 20, we obtain the following Corollary.

**Corollary 22** Let \((X, \leq)\) be partially ordered set and suppose there is a G-metric \(G\) on \(X\) such that \((X, G)\) is a G-complete. Suppose also that \(T: X \to X\) is an ICS mapping and \(F: X^3 \to X\) be a continuous mapping having the mixed monotone property on \(X\). Assume that there exists \(k \in [0,1]\) such that for \(x, y, z, a, b, c, u, v, w \in X\), with \(x \geq a \geq u, y \leq b \leq v, \) and \(z \geq c \geq w,\) one has

\[
G(TF(x, y, z), TF(a, b, c), TF(u, v, w) \leq k \max\{G(Tx, Ta, Tu), G(Ty, Tb, Tv), G(Tz, Tc, Tw)\}
\]

(36)

If there exist \(x_0, y_0, z_0 \in X\) be as in (8). Then \(F\) has a tripled fixed point in \(X\).

**Corollary 23** Let \((X, \leq)\) be partially ordered set and suppose there is a G-metric \(G\) on \(X\) such that \((X, G)\) is a G-complete. Suppose also that \(T: X \to X\) is an ICS mapping and \(F: X^3 \to X\) be a continuous mapping having the mixed monotone property on \(X\). Assume that there exists \(k \in [0,1]\) such that for \(x, y, z, a, b, c, u, v, w \in X\), with \(x \geq a \geq u, y \leq b \leq v, \) and \(z \geq c \geq w,\) one has

\[
G(TF(x, y, z), TF(a, b, c), TF(u, v, w) \leq k \max\{G(Tx, Ta, Tu) + G(Ty, Tb, Tv) + G(Tz, Tc, Tw)\}
\]

(37)

If there exist \(x_0, y_0, z_0 \in X\) be as in (8). Then \(F\) has a tripled fixed point in \(X\).

**Proof:** Note that

\[
G(Tx, Ta, Tu) + G(Ty, Tb, Tv) + G(Tz, Tc, Tw) \leq 3 \max\{G(Tx, Ta, Tu), G(Ty, Tb, Tv), G(Tz, Tc, Tw)\}
\]

(38)

Then, the proof follows from Corollary 22. ■

**Remark 24** Taking \(T = I_{G} \), the identity on \(X\), in Theorem 20; we get main result (Theorem 2.1) of Hassen et al. [13]. Therefore, Corollary 22 and 23 are generalization of Corollary 2.2 and 2.3 of Hassen et al. [13], respectively. ■

In the following theorem, we omit the continuity hypothesis of \(F\). We need the following definition.

**Definition 25** Let \((X, \leq)\) be a partially ordered set and \((X, G)\) be a G-metric. We say that \((X, G, \leq)\) is regular if the following conditions hold in \(X:\)

a) if a non-decreasing sequence \(x_n \to x\) in \(X\), then \(x_n \leq x, \forall n\).

b) if a non-increasing sequence \(y_n \to y\) in \(X\), then \(y_n \geq y, \forall n\).

c) if a non-decreasing sequence \(x_n \to x\) in \(X\), then \(x_n \leq x, \forall n\).

d) if a non-increasing sequence \(y_n \to y\) in \(X\), then \(y_n \geq y, \forall n\).

**Theorem 26** Let \((X, \leq)\) be partially ordered set and suppose there is a G-metric \(G\) on \(X\) such that \((X, G)\) is a G-complete. Suppose also that \(T: X \to X\) is an ICS mapping and \(F: X^3 \to X\) be a mapping having the mixed monotone property on \(X\). Assume that there exists \(\phi \in \Phi\), such that \(\phi(x, y, z, a, b, c, u, v, w) \in X\), with \(x \geq a \geq u, y \leq b \leq v, \) and \(z \geq c \geq w,\) one has (10). If there exist \(x_0, y_0, z_0 \in X\) be as in (8). Assume also \((X, G, \leq)\) is regular. Then \(F\) has a tripled fixed point in \(X\).

**Proof:** Following proof of Theorem 20 step by step, we can construct three sequences \([x_n],[y_n]\), and \([z_n]\) in \(X\) such that \(x_{n+1} = F(x_n, y_n, z_n)\), \(y_{n+1} = F(y_n, x_n, y_n)\), and \(z_{n+1} = F(z_n, z_n, y_n)\) with \(x_n \leq x_{n+1}, y_{n+1} \leq y_n,\) and \(z_n \leq z_{n+1}\). Then, \([Tx_n],[Ty_n]\), and \([Tz_n]\) are G-Cauchy sequences in \((X, G)\). Since \((X, G)\) is a G-complete, \([Tx_n],[Ty_n]\) and \([Tz_n]\) are convergent sequences. Since \(T\) is an ICS mapping, there exist
x, y, z ∈ X such that \{x_n\}, \{y_n\}, and \{z_n\} converge to x, y, and z, respectively. Since T is continuous, we have
\[
\begin{align*}
\lim_{n \to +\infty} T x_n &= T x, \\
\lim_{n \to +\infty} T y_n &= T y, \\
\lim_{n \to +\infty} T z_n &= T z.
\end{align*}
\] (39)

We remain to show that F has a fixed tripled point \((x, y, z)\) in X. To this aim, suppose that assumption "\((X, \mathcal{G}, \leq)\) is regular" holds. Since \(x_n\) and \(y_n\) are non-decreasing with \(x_n \to x\) and \(z_n \to z\) and also \(y_n\) is non-increasing with \(y_n \to y\). We have \(x_n \leq x\), \(y_n \geq y\), and \(z_n \leq z\), \(\forall n \). If for some \(n \geq 0\), \((x_n, y_n, z_n) = (x, y, z)\); that is, \(x_n = x\), \(y_n = y\), and \(z_n = z\); then \(x = x_n \leq x_{n+1} \leq x = x_n\). \(y = y_n \geq y_{n+1} \geq y = y_n\), and \(z = z_n\). This means that \(x_n = x_{n+1} = F(x_n, y_n, z_n)\), \(y_n = y_{n+1} = F(y_n, x_n, y_n)\) and \(z_n = z_{n+1} = F(z_n, y_n, z_n)\) is a tripled fixed point of F. Now, assume that, \(\forall n \geq 0\), \((x_n, y_n, z_n) \neq (x, y, z)\). Thus, \(\forall n \geq 0\),
\[
\max\{G(Tx, Tx, Tx), G(Ty, Ty, Ty), G(Tz, Tz, Tz)\} > 0.
\] (40)

From (10), we have
\[
\begin{align*}
G(TF(x, y, z), TF(x, y, z), TF(x, y, z)) &= G(T(F(x, y, z), TF(x, y, z), TF(x, y, z)) \leq \phi(\max\{G(Tx, Tx, Tx), G(Ty, Ty, Ty), G(Tz, Tz, Tz)\})
\end{align*}
\]
\[
\begin{align*}
G(Ty, Ty, Ty) &= G(T(Fy, Ty, Ty), TF(y, x, y), TF(y, x, y)) \\
&= \phi(\max\{G(Ty, Ty, Ty), G(Tx, Tx, Tx)\}) \\
&\leq \phi(\max\{G(Ty, Ty, Ty), G(Tx, Tx, Tx)\}) \\
&\leq \phi(\max\{G(Tz, Tz, Tz)\}) \leq \phi(\max\{G(Tz, Tz, Tz)\})
\end{align*}
\]
\[
\begin{align*}
G(TF(x, y, z), TF(x, y, z), TF(x, y, z)) &= G(T(F(z, y, x), TF(z, y, x), TF(z, y, x)) \\
&= \phi(\max\{G(Tz, Tz, Tz)\}) \leq \phi(\max\{G(Tz, Tz, Tz)\}) \\
&= \phi(\max\{G(Tz, Tz, Tz)\}) \leq \phi(\max\{G(Tz, Tz, Tz)\}) \\
&= \phi(\max\{G(Tz, Tz, Tz)\}) \leq \phi(\max\{G(Tz, Tz, Tz)\})
\end{align*}
\] (41)

Letting \(n \to +\infty\) in (41), using (40) in the fact that \(\phi(t) < t\) for all \(t \in (0, +\infty)\) and (39), the right-hand of all inequalities in (41) tends to 0, so we get that
\[
\begin{align*}
G(TF(x, y, z), TF(x, y, z), Tx) &= 0, \\
G(Ty, TF(y, x, y), TF(y, x, y)) &= 0, \\
G(TF(x, y, z), TF(x, y, z), Ty) &= 0.
\end{align*}
\]
This means that \(TF(x, y, z) = Tx\), \(TF(y, x, y) = Ty\), and \(TF(z, y, x) = Tz\). Since T is injective, it follows that
\[
F(x, y, z) = x, \quad F(y, x, y) = y, \quad F(z, y, x) = z.
\]
Thus, we proved that F has a tripled fixed point in X. This completes the proof.

**Remark 27** Results similar to Corollary 21, 22 and 23 omitting the continuity hypotheses of F and involving hypotheses \((X, \mathcal{G}, \leq)\) is regular corresponding to Theorem 26 can also be derived. Due to repetition, the details are avoided.

### 4. Uniqueness of a Tripled Fixed Point

Now, we shall prove the uniqueness of a tripled fixed point. For a product \(X^3 = X \times X \times X\) of a partial ordered set \((X, \leq)\), we define a partial ordering in the following way: for all \((x, y, z), (u, v, w) \in X^3\),
\[
(x, y, z) \leq (u, v, w) \iff x \leq u, \quad y \geq v, \quad z \leq w.
\] (42)
We say that \((x, y, z)\) and \((u, v, w)\) are comparable if
\[
\begin{align*}
&(x, y, z) \leq (u, v, w), \\
&\text{Or} \quad (u, v, w) \leq (x, y, z).
\end{align*}
\] (43)

Also, we say that \((x, y, z)\) is equal to \((u, v, w)\) if and only if \(x = u, \quad y = v\), and \(z = w\).

**Theorem 28** In addition to hypotheses of Theorem 28, suppose that for all \((x, y, z), (u, v, w) \in X^3\), there exists \((a, b, c) \in X^3\) such that
\[
(F(a, b, c), F(b, a, b), F(c, b, a))
\]
is comparable to
\[
(F(x, y, z), F(y, x, y), F(z, y, x), )
\]
and
\[
(F(u, v, w), F(v, u, v), F(w, v, u))
\]
Then F has a unique triple fixed point \((x, y, z)\).

**Proof:** Due to Theorem 20, the set of tripled fixed points of F is not empty. Suppose \((x, y, z)\) and \((u, v, w)\), are triple fixed points of the mapping \(F : X^3 \to X\) such that \((x, y, z) \neq (u, v, w)\); that is,
\[
\begin{align*}
F(x, y, z) &= x, \quad F(u, v, w) = u, \\
F(y, x, y) &= y, \quad F(v, u, v) = v, \\
F(z, y, x) &= z, \quad F(w, v, u) = w.
\end{align*}
\] (44)
We shall show that \((x, y, z)\) and \((u, v, w)\) are equal. By assumption, there exists \((a, b, c) \in X^3\) such that
\[
(F(a, b, c), F(b, a, b), F(c, b, a))
\]
is comparable to
\[
(F(x, y, z), F(y, x, y), F(z, y, x), )
\]
and
\[
(F(u, v, w), F(v, u, v), F(w, v, u))
\]
Put \(a_0 = a, b_0 = b, c_0 = c\) and choose \(a_1, b_1, c_1\) in X such that
\[
a_1 = F(a_0, b_0, c_0), \quad b_1 = F(b_0, a_0, b_0), \quad c_1 = F(c_0, b_0, a_0).
\]
Thus, we can define three sequences \(\{a_n\}, \{b_n\}, \{c_n\}\) as
\[
\begin{align*}
a_n &= F(a_{n-1}, b_{n-1}, c_{n-1}), \\
b_n &= F(b_{n-1}, a_{n-1}, b_{n-1}), \\
c_n &= F(c_{n-1}, b_{n-1}, c_{n-1}).
\end{align*}
\] (45)
for any $n \geq 1$. Further set $x_0 = x$, $y_0 = y$, $z_0 = z$ and $u_0 = u$, $v_0 = v$, $w_0 = w$, and on the same way define the sequences $\{x_n\}, \{y_n\}$, and $\{z_n\}$ and $\{u_n\}, \{v_n\}$, and $\{w_n\}$. Then, it is easy that

$$
\begin{align*}
x_n &= F(x, y, z), \\
y_n &= F(u, v, w), \\
z_n &= F(z, x, y),
\end{align*}
$$

is comparable to

$$
(F(x, y, z), F(y, x, y), F(x, y, x)) = (x_1, y_1, z_1) = (x, y, z)
$$

Similarly, we have:

$$
(G(Tx, Ty, Tb_{n+1}) = G(TF(x, y, z), TF(x, y, z), TF(a_{m}, b_{n}, c_{n})),
$$

$$
\leq \phi(\max\{G(Tx, Ty, Ta_{n}), G(Ty, Ty, Tb_{n}), G(Tz, Tz, Tc_{n})\})
$$

Similarly, we have;

$$
(G(Ty, Ty, Tb_{n+1})
$$

$$
= G(TF(x, y, z), TF(x, y, z), TF(b_{n}, a_{m}, b_{n})),
$$

$$
\leq \phi(\max\{G(Ty, Ty, Tb_{n}), G(Tx, Tx, Ta_{n}), G(Tz, Tz, Tc_{n})\})
$$

Similarly, we have;

$$
(G(Tz, Tz, Tc_{n+1})
$$

$$
= G(TF(x, y, z), TF(z, x, y), TF(c_{n}, b_{n}, a_{m})),
$$

$$
\leq \phi(\max\{G(Tz, Tz, Tc_{n}), G(Ty, Ty, Tb_{n}), G(Tz, Tz, Tc_{n})\})
$$

It follows from (46) and (47) that

$$
\max\{G(Tx, Tx, Ta_{n+1}), G(Ty, Ty, Tb_{n+1}), G(Tz, Tz, Tc_{n+1})\}
$$

$$
\leq \phi(\max\{G(Tx, Tx, Ta_{n}), G(Ty, Ty, Tb_{n}), G(Tz, Tz, Tc_{n})\})
$$

Therefore, for each $n \geq 1$, we have

$$
\max\{G(Tx, Tx, Ta_{n}), G(Ty, Ty, Tb_{n}), G(Tz, Tz, Tc_{n})\}
$$

$$
\leq \phi^n(\max\{G(Tx, Tx, Ta_{0}), G(Ty, Ty, Tb_{0}), G(Tz, Tz, Tc_{0})\})
$$

$$
\begin{align*}
\lim_{n \to \infty} \phi^n(t) = 0 \quad &\text{for each } t > 0. \quad \text{Thus, from (49), we have}
\end{align*}

$$
\lim_{n \to \infty} \max\{G(Tx, Tx, Ta_{n}), G(Ty, Ty, Tb_{n}), G(Tz, Tz, Tc_{n})\} = 0. \quad (50)
$$

This yields that

$$
\lim_{n \to \infty} G(Tx, Tx, Ta_{n}) = 0,
$$

$$
\lim_{n \to \infty} G(Ty, Ty, Tb_{n}) = 0,
$$

$$
\lim_{n \to \infty} G(Tz, Tz, Tc_{n}) = 0, \quad (51)
$$

Similarly, we can prove the following statement:

**Theorem 29** In addition to hypotheses of Theorem 26, suppose that for all $(x, y, z)$, $(a, b, c) \in X^3$, there exists $(a, b, c) \in X^3$ such that

$$
(F(a, b, c), F(b, a, b), F(c, b, a))
$$

is comparable to

$$
(F(x, y, z), F(x, y, z), F(z, x, y))
$$

and

$$
(F(u, v, w), F(u, v, w), F(w, v, w)).
$$

Then $F$ has a unique triple fixed point $(x, y, z)$. ■

5. EXAMPLES

In this section, we state some examples showing that our results are effective.

**Example 30** As in Example 18 (a), define $T: X \to X$ be defined by $Tx = ln(x) + 1, \forall x \in X$. Obviously, $T$ is an ICS mapping. Define $\phi: \mathbb{R}^+ \to \mathbb{R}^+$ be defined by $\phi(t) = t/2, \forall t > 0$, then $\phi \in \Phi$. Taking $x, y, z, a, b, c, u, v, w \in X$, with $x \geq a \geq b \geq c \geq w$, we have

$$
G(Tx, Ty, Tz, TF(a, b, c), TF(u, v, w))
$$

$$
= G(T \left( \frac{\sqrt{3v}}{\sqrt{y}} \right)^2, T \left( \frac{\sqrt{3v}}{\sqrt{b}} \right)^2, T \left( \frac{\sqrt{3v}}{\sqrt{v}} \right)^2)
$$

$$
= G \left( \ln \left( \frac{\sqrt{3v}}{\sqrt{y}} \right)^2 \right) + 1, \ln \left( \frac{\sqrt{3v}}{\sqrt{b}} \right)^2 + 1,
$$

$$
\ln \left( \frac{\sqrt{3v}}{\sqrt{v}} \right)^2 + 1
$$

$$
= \ln \left( \frac{\sqrt{3v}}{\sqrt{y}} \right)^2 - \ln \left( \frac{\sqrt{3v}}{\sqrt{b}} \right)^2 + \ln \left( \frac{\sqrt{3v}}{\sqrt{v}} \right)^2
$$

$$
+ \ln \left( \frac{\sqrt{3v}}{\sqrt{y}} \right)^2 - \ln \left( \frac{\sqrt{3v}}{\sqrt{b}} \right)^2 + \ln \left( \frac{\sqrt{3v}}{\sqrt{v}} \right)^2
$$

$$
\leq \frac{1}{12} \left( |\ln(x) - \ln(a)| + |\ln(a) - \ln(u)| + |\ln(u) - \ln(x)| \right)
$$
by also the hypotheses of Theorem 28 hold.

So Theorem 20 can be applied to this example to mapping. Set

\[
\frac{1}{12}g(Tx, Ta, Tu) + \frac{1}{12}g(Tz, Tc, Tw) + \frac{1}{12}g(Tw, Tb, Tv)
\]

\[
\leq \frac{1}{6}(g(Tx, Ta, Tu) + g(Tz, Tc, Tw))
\]

\[
\leq \frac{1}{2}\max\{g(Tx, Ta, Tu), g(Tz, Tc, Tw), g(Tw, Tb, Tv)\}
\]

\[
\phi(\max\{g(Tx, Ta, Tu), g(Ty, Tb, Tv), g(Tz, Tc, Tw)\})
\]

This is the contractive condition (10). Evidently, for every \((x, y, z), (u, v, w) \in X^3\), there always exists a point \((a, b, c) \in X^3\) that is comparable to \((x, y, z)\) and \((u, v, w)\). So Theorem 20 can be applied to this example to conclude that \(F\) has a unique tripled fixed point \((8,8,8)\), since also the hypotheses of Theorem 28 hold.

Example 31 As in Example 18, let \(T : X \to X\) by \(Tx = \ln(x) + 1, \forall x \in X\). Obviously, \(T\) is an ICS mapping. Set \(\phi(t) = \frac{2t}{3} \in \Phi\). Taking \(x, y, z, a, b, c, u, v, w \in X\), with \(x \geq a \geq u, y \geq b \geq v, \) and \(z \geq c \geq w\),

\[
G(TF(x, y, z), TF(a, b, c), TF(u, v, w))
\]

\[
= G\left(T \left(8^{\frac{1}{3}}\right)T \left(8^{\frac{1}{3}}\right)T \left(8^{\frac{1}{3}}\right)\right)
\]

\[
= G\left(\ln \left(8^{\frac{1}{3}}\right) + 1, \ln \left(8^{\frac{1}{3}}\right) + 1, \ln \left(8^{\frac{1}{3}}\right) + 1\right)
\]

\[
= \frac{1}{3}\left[\ln(\frac{8}{8}) - \ln(\frac{8}{8})\right] + [\ln(\frac{8}{8}) - \ln(\frac{8}{8})]
\]

\[
= \frac{1}{3}\left[\ln(x) - \ln(y) - (\ln(a) - \ln(b))\right] + [\ln(a) - \ln(b) - \ln(\alpha) - \ln(\beta)]
\]

\[
\leq \frac{1}{3}\left[\ln(x) - \ln(\alpha)\right] + [\ln(\alpha) - \ln(\beta)]
\]

\[
= \phi(\max\{G(Tx, Ta, Tu), g(Ty, Tb, Tv), g(Tz, Tc, Tw)\})
\]

This is the contractive condition (10). Evidently, for every \((x, y, z), (u, v, w) \in X^3\), there always exists a point \((a, b, c) \in X^3\) that is comparable to \((x, y, z)\) and \((u, v, w)\). So Theorem 20 can be applied to this example to conclude that \(F\) has a unique tripled fixed point \((8,8,8)\), since also the hypotheses of Theorem 28 hold.

Now, we give a simple example which shows that if \(T\) is not an ICS mapping then the conclusion of Theorem 20 and 26 fail.

Example 32 Let \(X = \mathbb{R}\) and define \(G : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}\) by \(\forall x, y, z \in X\),

\[
G(x, y, z) = \max\{|x - y|, |x - z|, |y - z|\}
\]

Let \(\leq\) be usual order. Then, \((X, G)\) is a G-complete G-space. Let \(F : X^3 \to X\) be given by

\[
F(x, y, z) = 2x - y + 1, \forall x, y, z \in X.
\]

It is clear that \(F\) is continuous and has the mixed monotone property. Moreover, taking \(x_0 = z_0 = 1\) and \(y_0 = 0\), we have

\[
F(x_0, y_0, z_0) = F(1,0,1) = 3 > 1 = x_0,
\]

\[
F(y_0, y_0, y_0) = F(0,1,0) = -1 < 0 = y_0,
\]

\[
F(z_0, y_0, y_0) = F(1,1,1) = 3 > 1 = z_0,
\]

it is condition (8). Let \(T : X \to X\) be defined by \(Tx = x, \forall x \in X\). Then \(T\) is not an ICS mapping. It is obvious that the condition (10) holds for any \(\Phi \in \Phi\). However, \(F\) has no tripled fixed point.

Example 33 As in Example 2.5 of [13], let \(T : X \to X\) be given by \(Tx = x, \forall x \in X\). Obviously, \(T\) is an ICS mapping. The mapping \(F : X^3 \to X\) has unique tripled fixed point \((0,0,0)\).

Example 34 As in Example 2.6 of [13], let \(T : X \to X\) be given by \(Tx = x, \forall x \in X\). Obviously, \(T\) is an ICS mapping. The mapping \(F : X^3 \to X\) has unique tripled fixed point \((0,0,0)\).

5. Conclusion

In this paper, we established some tripled fixed point theorems for mappings having mixed monotone property under nonlinear type contractions depended on another function \(T : X \to X\) (where \(T\) is an ICS mapping) in the framework of a G-complete space \(X\) enclosed with partial order. Our results are generalized, improved and extended some well-known results in the literature. These results are extensions of results in [13] to the case triple fixed points depending on another function. Inequality (10) does not reduce to any metric inequality with the metric \(d_1\), [this metric is given by (2)]. Hence our theorems do not reduce to fixed point problems in the corresponding metric space \((X, d_1)\). Also, in all Theorem 20 (Theorem 26) is genuinely different to Theorem 2.1 (Theorem 3.4) of Hassen et al. [13]. If mapping \(T : X \to X\)
is not an ICS mapping then the conclusion of main results (Theorems 20 and 26) fail (see Example 31). Also, presented examples are showing that our results are real generalization of known ones in triple fixed point theory. Our results may be the motivation to other authors for extending and improving these results to be suitable tools for their applications.

COMPETING INTERESTS

No conflict of interest was declared by the authors.

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