A Note on Multivariate Lyapunov-Type Inequality

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ABSTRACT

We transfer the recent obtained result of univariate Lyapunov-type inequality for third order differential equations to the multivariate setting of a shell via the polar method. Our result is better than the result of Anastassiou [Appl. Math. Letters, 24 (2011), 2167-2171] for third order partial differential equations.

Keywords: Lyapunov-type inequality; Shell; Third order.

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1. INTRODUCTION AND MAIN RESULT

The Lyapunov inequality and many of its generalizations play a key role in the study of oscillation theory, disconjugacy, eigenvalue problems, and numerous other applications for the theories of differential and difference equations. Up to now, the Lyapunov-type inequalities have been studied extensively such as [1,2,6,9]. However, there are not so many results for partial differential equations or systems except for [3,4] or [5].

Here, we give some notation for constructing the theoretical background given by Anastassiou [3] who was the first interested in the problem of finding on the multivariate Lyapunov-type inequalities in the literature:

Suppose that $A$ be a spherical shell $\subseteq \mathbb{R}^N$ for $N>1$, i.e. $A = B(0, R_1) - B(0, R_2)$ for $0 < R_1 < R_2$, where the ball $B(0, R) = \{ x \in \mathbb{R}^N : |x| < R \}$

for $R > 0$ and $|\cdot|$ is the Euclidean norm. We also suppose that

$S^{N-1} = \{ x \in \mathbb{R}^N : |x| = 1 \}$

is the unit sphere in $\mathbb{R}^N$ with surface area

$\omega_N = \frac{2\pi^{N/2}}{\Gamma(N/2)}$,

i.e.

$\int_{S^{N-1}} d\omega \frac{2\pi^{N/2}}{\Gamma(N/2)}$

where $\Gamma$ is the gamma function. It is easy to see that every $x \in \mathbb{R}^N - \{0\}$ has a unique representation of the form $x = r\omega$, where $r = |x| > 0$ and $\omega = \frac{x}{r} \in S^{N-1}$ [8, pp. 149-150]. Thus, $\mathbb{R}^N - \{0\}$ may be regarded as the Cartesian product $\lambda = [R_1, R_2] \times S^{N-1}$. Therefore, we have

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\[
\int_a F(x)ds = \int_a^b \left( \int_0^{\theta} f(r\omega) r^{n-1} dr \right) d\omega
\] (5)

for \( F \in C(\overline{A}) \). Here, we deal with the partial differential equations involving radial derivatives of functions on \( \overline{A} \), using the polar coordinates \( r, \omega \). If \( f \in C(\overline{A}) \), for \( n \in \mathbb{N} \), then \( f(\omega) \in C(\{R_i, R_j\}) \) for a fixed \( \omega \in S^{n-1} \).

Recently, by using the result of Çakmak [6], Anastassiou [3] obtained the following result.

**Theorem A.** Suppose that \( n \in \mathbb{N}, \ n \geq 2 \) and \( q \in C(\overline{A}) \). If \( f \in C(\overline{A}) \) is a solution of the following partial differential equations

\[
\frac{\partial^2 f(x)}{\partial r^2} + q(x)f(x) = 0, \quad \forall x \in \overline{A}.
\] (6)

with the boundary value conditions

\[
f(\partial B(0, R_i)) = f(\partial B(0, R_j)) = \cdots = f(\partial B(0, R_{n-1})) = f(\partial B(0, R_n)) = 0
\] (7)

where \( R_i = t_1 < t_2 < \cdots < t_{n-1} < t_n = R_n \), and \( f(\omega) \neq 0 \), \( \forall \omega \in S^{n-1} \), for any \( t \in (t_i, t_{i+1}) \), \( k = 1, 2, \ldots, n-1 \), then the following inequality

\[
\int_{\partial B(0, R_i)} |f(x)|ds > \frac{4}{(n-2)!} \frac{R_i^{n-1}}{(n-1)^{n-1}(R_n - R_i)^{n-1}} \frac{2\pi^{1/2}}{\Gamma(n/2)}
\] (8)

holds.

In 1907, Lyapunov [7] established the first Lyapunov inequality

\[
\int_{\partial B(0, R_i)} |f(x)|ds > \frac{4}{c-a}.
\] (9)

if

\[
x^*(t) + q(t)x(t) = 0
\] (10)

has a real solution \( x(t) \) satisfying the boundary value conditions

\[
x(a) = x(c) = 0
\] (11)

for \( x(t) \neq 0 \) for \( t \in (a, c) \).

Since the appearance of Lyapunov’s fundamental paper, various proofs and generalizations or improvements have appeared in the literature.

More recently, Aktaş et al. [1] obtained the following Lyapunov-type inequality for third order differential equations

\[
x^*(t) + q(t)x(t) = 0,
\] (12)

where \( q \in C([a, c]) \), with the boundary value conditions

\[
x(a) = x(b) = x(c) = 0
\] (13)

for \( x(t) \neq 0 \) for \( t \in (a, b) \cup (b, c) \).

**Theorem B.** If the equation (12) has a solution \( x(t) \) satisfying the boundary value conditions (13), then the following inequality

\[
\int_{\partial B(0, R_i)} |f(x)|ds > \frac{16}{(c-a)^2}
\] (14)

holds.

Now, motivated by the recent results of Anastassiou [3], we transfer the univariate inequality (14) in Theorem B to the multivariate setting of a shell via the polar method.

**Theorem 1.** Suppose that \( q \in C(\overline{A}) \). If \( f \in C(\overline{A}) \) is a solution of the following third order partial differential equations

\[
\frac{\partial^3 f(x)}{\partial r^3} + q(x)f(x) = 0, \quad \forall x \in \overline{A}.
\] (15)

with the boundary value conditions

\[
f(\partial B(0, R_i)) = f(\partial B(0, R_j)) = \cdots = f(\partial B(0, R_{n-1})) = f(\partial B(0, R_n)) = 0
\] (16)

where \( R_i < R_j < R_n \), and \( f(\omega) \neq 0 \), \( \forall \omega \in S^{n-1} \), for any \( t \in (R_i, R_j) \cup (R_j, R_n) \), then the following inequality

\[
\int_{\partial B(0, R_i)} |f(x)|ds > \frac{16R_i^{n-1}}{(R_n - R_i)^{n-1}} \frac{2\pi^{1/2}}{\Gamma(n/2)}
\] (17)

holds.

**Proof.** One can rewrite (15) as

\[
\frac{\partial^3 f(r\omega)}{\partial r^3} + q(r\omega)f(r\omega) = 0, \quad \forall (r, \omega) \in [R_i, R_n] \times S^{n-1}.
\] (18)

where \( q(\omega) \in C([R_i, R_n]), \forall \omega \in S^{n-1} \), such that the boundary value conditions

\[
f(R_i\omega) = f(R_j\omega) = f(R_n\omega) = 0
\] (19)

for \( \forall \omega \in S^{n-1} \). In addition, \( f(\omega) \neq 0 \) holds for any \( r \in (R_i, R_j) \cup (R_j, R_n) \) and \( \forall \omega \in S^{n-1} \). Thus, from inequality (14), we get
\[
\frac{16}{(R_{e}-R_{i})} \leq \int_{R_{i}}^{R_{e}} |p(r)| dr = \\
= \int_{R_{i}}^{R_{e}} r^{-1} |p(r)| dr \leq \left( \int_{R_{i}}^{R_{e}} r^{-1} |p(r)| dr \right) R_{e}^{-1} \\
\] 
(20)

for a fixed \( \omega \in S^{N-1} \). Therefore, we have the following inequality

\[
\int_{R_{i}}^{R_{e}} r^{-1} |p(r)| dr > \frac{16R_{e}^{N-1}}{(R_{e}-R_{i})} \\
\] 
(21)

for \( \forall \omega \in S^{N-1} \) and

\[
\int_{S^{N-1}} \left( \int_{R_{i}}^{R_{e}} r^{-1} |p(r)| dr \right) d\omega > \left( \int_{R_{i}}^{R_{e}} r^{-1} |p(r)| dr \right) R_{e}^{-1} \left( \frac{2\pi^{N/2}}{\Gamma(N/2)} \right) \\
\] 
(22)

which by (5), proves the inequality (17).

**Remark 1.** It is easy to see that the inequality (17) is better than the inequality (8) with \( n=3 \) in Theorem A given by Anastassiou [3] in the sense that (8) with \( n=3 \) follows from (17), but not conversely.

**CONFLICT OF INTEREST**

No conflict of interest was declared by the authors.

**REFERENCES**


