Weighted Approximation by the 
$q$–Szász–Schurer–Beta Type Operators

İsmet YÜKSEL$^1$, Ülkü DINLEMEZ$^2,*$

$^1,2$Gazi University, Faculty of Science, Department of Mathematics, Teknikokullar, 06500, Ankara, Turkey

ABSTRACT

In this study, we investigate approximation properties of a Schurer type generalization of $q$–Szász-beta type operators. We estimate the rate of weighted approximation of these operators for functions of polynomial growth on the interval $[0, \infty)$.

2000 Math. Subject Classification: 41A25,41A36.

Key words and phrases: $q$–Szász–Schurer–beta operators, $q$–mixed operators, weighted approximation, rates of approximation.

1. INTRODUCTION

In [1], based on $q$–integer and $q$–binomial coefficients, firstly, Lupaş introduced a $q$–analogue of the Bernstein operators. After then several interesting generalization about $q$–calculus were given in [2]–[13]. Recently, in [14], Dinlemez studied convergence of the $q$–Stancu-Szász-beta type operators. Our aim is to study on Schurer type generalization of $q$–Szász-beta type operators. We use without further explanation the basic notations and formulas, from the theory of $q$–calculus as set out in [15]–[19]. Let $A > 0$ and $f$ be a real valued continuous function defined on the interval $[0, \infty)$. For $0 < q \leq 1$, $q$–Szász–Schurer-beta type operators are defined as

$$S_{n,p,q}(f, x) = \sum_{k=0}^{\infty} s_{n,p,k}^q(x) \int_0^\infty b_{n,p,k}^q(t) f(t) d_q(t),$$

(1)

where

$$s_{n,p,k}^q(x) = [(n + p)q]^k a^{n+p}_k x^k, \\
\text{and} \\
b_{n,p,k}^q(x) = \frac{q^k a^{n+p}_k (1+x)^{n+p+k}}{a^{n+p+k+1}(1+x)^{n+p+k+1}}.$$

If we write $p = 0$ in (1), then the operators $S_{n,p,q}$ are reduced to $q$–Szász-beta type operators studied in [10,11] and [14]. If we write $p = 0$ and $q = 1$ in (1), then the operators $S_{n,p,q}$ are reduced to Szász-beta type operators given in [20] – [23].

*Corresponding author, e-mail: ulku@gazi.edu.tr
2. MOMENT ESTIMATION

For the sake of shortness, the notation $F_s^{-1}(m) = \prod_{i=0}^{s}[m-i]_q$ will be used throughout the article. In the following lemma, lemma is similarly to the proof given in [10,11]. So proof of the following lemma is omitted.

**Lemma 1.** $e_m(t) = t^m, m = 0, 1, 2, 3$ and 4. Then, we get

(i) $S_{n,p,q}(e_0,x) = 1$,

(ii) $S_{n,p,q}(e_1,x) = \frac{[n+p]_q}{q^{\nu}F_s(n+p-1)}x + \frac{1}{q^{\nu}F_s(n+p-1)}$,

(iii) $S_{n,p,q}(e_2,x) = \frac{[n+p]_q}{q^{\nu}F_s(n+p-1)}x^2 + \frac{\nu [\nu+q][n+p]_q}{q^{2\nu}F_s(n+p-1)}x + \frac{[2]_q}{q^{2\nu}F_s(n+p-1)}$,

(iv) $S_{n,p,q}(e_3,x) = \frac{[n+p]_q}{q^{\nu}F_s(n+p-1)}x^3 + \frac{\nu [\nu+q][n+p]_q}{q^{2\nu}F_s(n+p-1)}x^2 + \frac{2[\nu]_q}{q^{3\nu}F_s(n+p-1)}x + \frac{2[2]_q}{q^{3\nu}F_s(n+p-1)}$,

(v) $S_{n,p,q}(e_4,x) = \frac{[n+p]_q}{q^{\nu}F_s(n+p-1)}x^4 + \frac{\nu [\nu+q][n+p]_q}{q^{2\nu}F_s(n+p-1)}x^3 + \frac{3\nu^2 [\nu+q][n+p]_q}{q^{3\nu}F_s(n+p-1)}x^2 + \frac{6\nu [\nu]_q}{q^{4\nu}F_s(n+p-1)}x + \frac{2[\nu]_q}{q^{4\nu}F_s(n+p-1)}$.

To obtain our main results we need to compute second and fourth moments.

**Lemma 2.** Let $q \in (0,1)$ and $n > 4$. Then, we have the following inequalities

(i) $S_{n,p,q}((t−x)^2,x) \leq \left(\frac{1−q^3}{q^2} + \frac{11}{q^{4\nu}F_s(n+p-1)}\right)x(x+1) + \frac{2}{q^{4\nu}F_s(n+p-1)}$ and

(ii) $S_{n,p,q}((t−x)^4,x) \leq \left(\frac{1}{q^{6\nu}F_s(n+p-1)} + \frac{1296}{q^{6\nu}F_s(n+p-1)}\right)x^4 + x^3 + x^2 + x + 1$.

**Proof of (i).** From linearity of $S_{n,p,q}$ operators and Lemma 1, we have the second moment as

\[
S_{n,p,q}((t−x)^2,x) = S_{n,p,q}(t^2,x) − 2x S_{n,p,q}(t,x) + x^2 S_{n,p,q}(1,x) = \frac{2\nu [n+p]_q}{q^{4\nu}F_s(n+p-1)}x^2 + \frac{2[\nu]_q}{q^{3\nu}F_s(n+p-1)}x + \frac{2[2]_q}{q^{3\nu}F_s(n+p-1)}
\]

(2)

In (2), using the inequality $[n+p−s]_q \leq [n+p]_q$ for $s > 0$ and ignoring some negative terms, we obtain

\[
S_{n,p,q}((t−x)^2,x) \leq \frac{1}{q^{4\nu}F_s(n+p-1)}(2\nu [n+p]_q − 2q\nu [n+p−2]_q)x^2 + \frac{2[\nu]_q}{q^{3\nu}F_s(n+p-1)}x + \frac{2[2]_q}{q^{3\nu}F_s(n+p-1)}
\]

(3)

In (3), using the inequality

$[n+p]_q \leq [s]_q + q^3[n+p−s]_q$ for $s > 0$,

we get

\[
S_{n,p,q}((t−x)^2,x) \leq \frac{1}{q^{4\nu}F_s(n+p-1)}(2\nu [n+p]_q + 2q\nu [n+p−2]_q + 2q^3[n+p−2]_q)x^2 + \frac{2[\nu]_q}{q^{3\nu}F_s(n+p-1)}x + \frac{2[2]_q}{q^{3\nu}F_s(n+p-1)}
\]

(4)

And the proof of (i) of the Lemma 2 is now finished.

**Proof of (ii).** From linearity of $S_{n,p,q}$ operators and Lemma 1, we write the fourth moment as

$S_{n,p,q}((t−x)^4,x)$.
= S_{n,p,q}(t^4,x) - 4xS_{n,p,q}(t^3,x) + 6x^2S_{n,p,q}(t^2,x) - 4x^3S_{n,p,q}(t,x) + x^4S_{n,p,q}(1,x)

= \frac{C_1(n,p,q)}{q^{2n}F_2(n+p-1)} x^2 + \frac{C_2(n,p,q)}{q^{2n}F_2(n+p-1)} x + \frac{C_3(n,p,q)}{q^{2n}F_2(n+p-1)}

(5)

Where

\begin{align*}
C_1(n,p,q) &= \frac{\lceil n + p \rceil_4^2}{4} - 4q^4[(n + p)\lceil n + p - 4 \rceil_3 - 6q^4[n + p]_3[n + p - 3]_4 + 4q^8[n + p]_3[n + p - 4]_4 \\
&- 4q^{16}[n + p]_4[n + p - 4]_4 + q^{20}F_2(n + p - 1) - 4q^8[n + p]_3[n + p - 4]_4 \\
&- 4q^{16}[n + p]_4[n + p - 4]_4 + q^{20}F_2(n + p - 1) - 4q^8[n + p]_3[n + p - 4]_4 + 4q^{16}[n + p]_4[n + p - 4]_4.
\end{align*}

\begin{align*}
C_2(n,p,q) &= \left( \frac{1}{12}q\lceil n \rceil_2 + q\lceil 2 \rceil_2\lceil 3 \rceil_4 \right) 2q^{16} + q^{18}\lceil n \rceil_3 + 4q^{18}\lceil n \rceil_3.
\end{align*}

Now, we find upper boundaries for the coefficients $C_i(n,p,q)$, $i = 1, 2, 3$ and 4. Again using the inequalities $[n + p + s]_4 \leq [n + p + s]_q$, we write

\begin{align*}
C_1(n,p,q) &= \frac{\lceil n + p \rceil_4^2}{4} + 4q^{16}[n + p]_4[n + p - 4]_4 + 4q^{18}\lceil n + p \rceil_4 + 4q^{18}\lceil n + p \rceil_4, \\
C_2(n,p,q) &\leq 4n + [n + p]_4.
\end{align*}

(6)

(7)

(8)

(9)

and

\begin{align*}
C_2(n,p,q) &= 24.
\end{align*}

(10)

Combining among (5) and (10), we get

\begin{align*}
S_{n,p,q}(x + t) - 4xS_{n,p,q}(x) + 6x^2S_{n,p,q}(x) - 4x^3S_{n,p,q}(x) + x^4S_{n,p,q}(1,x)
\end{align*}

\begin{align*}
&\leq \frac{8(q^{16} - q^{18})[n + p + 4]_4 + 4q^{18}[n + p + 4]_4}{q^{20}[n + p + 4]_4} x^4 \nonumber \\
&+ \frac{400[n + p + 4]_4 + 12n + p + 4]_4 + 48[n + p + 4]_4 + 64}{[n + p + 4]_4} x^3 \\
&+ \frac{4080(n + p + 4]_4 + 48(n + p + 4]_4 + 64)}{q^{16} + q^{18}} x^2 \\
&+ \frac{24}{q^{16} + q^{18}} x + \frac{12596}{q^{20}[n + p + 4]_4} x + \frac{q^{20}[n + p + 4]_4}{q^{20}[n + p + 4]_4} x + \frac{1}{q^{20}[n + p + 4]_4}.
\end{align*}

\begin{align*}
&\leq \frac{8(q^{16} - q^{18})[n + p + 4]_4 + 4q^{18}[n + p + 4]_4}{q^{20}[n + p + 4]_4} x^4 + \frac{q^{20}[n + p + 4]_4}{q^{20}[n + p + 4]_4} x^3 + \frac{q^{20}[n + p + 4]_4}{q^{20}[n + p + 4]_4} x + \frac{1}{q^{20}[n + p + 4]_4}
\end{align*}

\begin{align*}
&\leq (x^4 + x^3 + x^2 + x + 1).
\end{align*}

And the proof of (ii) of Lemma 2 is now finished.

3. LOCAL APPROXIMATION

Now, $C_B[0, \infty)$ denotes the space of bounded continuous functions with the norm $||f||_B = \sup_x(|f(x)|: x \in [0, \infty))$. We denote the first modulus of continuity on the finite interval $[0, b], b > 0,$

\begin{align*}
\omega_0(b) &= \sup_{0 < h, x \in [0, b]} |f(x + h) - f(x)|.
\end{align*}

The Peetre’s $K$–functional is defined by

\begin{align*}
K_2(f, \delta) &= \inf \{ ||f - g||_B + \delta||g''||_B : g \in W_2^2, \delta > 0 \},
\end{align*}

where $W_2^2 = \{ g \in C_B[0, \infty) : g', g'' \in C_B[0, \infty) \}.$

By [18, p. 177, Theorem 2.4], there exists a positive constant $C$ such that

\begin{align*}
K_2(f, \delta) \leq C \omega_2(f, \sqrt{\delta}),
\end{align*}

(12)
where $\omega_2(f, \sqrt{\delta}) = \sup_{0<h<\pi/2} \sup_{x\in[0,\pi]} |f(x+2h) - 2f(x+h) + f(x)|$.

Thus we are ready to give direct results. The following lemma is routine and its proof is omitted.

**Lemma 3.** Let

$$\tilde{S}_{n,p,q}(f,x) = S_{n,p,q}(f,x) - f \left( \frac{[n+p]_q}{q^{\frac{(n+p-1)}{2}}} x + \frac{1}{q^{\frac{n-(n+p-1)}{2}}} \right) + f(x).$$

(13)

Then, the operators $\tilde{S}_{n,p,q}$ satisfy the following assertions:

(i). $\tilde{S}_{n,p,q}(1,x) = 1$,

(ii). $\tilde{S}_{n,p,q}(t,x) = x$,

(iii). $\tilde{S}_{n,p,q}(t-x,x) = 0$.

**Lemma 4.** Let $q \in (0,1)$ and $n > 1$. Then for every $x \in [0,\infty)$ and $f'' \in C_0[0,\infty)$, we have the following inequality

$$|\tilde{S}_{n,p,q}(f,x) - f(x)| \leq \|f''\|_B \delta_{n,p,q}(x).$$

where

$$\delta_{n,p,q}(x) = \left( \frac{2(1-q^n)}{q^n} + \frac{28}{q^{n-(n+p-1)}} \right) x(1 + x) + \frac{2}{q^{n-(n+p-1)}}$$

**Proof.** Using Taylor’s expansion

$$f(t) = f(x) + (t-x)f'(x) + \int_x^t (t-u)f''(u)du$$

and from Lemma 3, we obtain

$$\tilde{S}_{n,p,q}(f,x) - f(x) = S_{n,p,q}\left( \int_x^t (t-u)f''(u)du, x \right).$$

Then, using the Lemma 2 (i) and the inequality

$$\left| \int_x^t (t-u)f''(u)du \right| \leq \|f''\|_B \frac{(t-x)^2}{2},$$

we get

$$|\tilde{S}_{n,p,q}(f,x) - f(x)| \leq \|f''\|_B \frac{(t-x)^2}{2} \leq \|f''\|_B \frac{(t-x)^{n-p}}{2} \left( \frac{2(1-q^n)}{q^n} + \frac{28}{q^{n-(n+p-1)}} \right) x(1 + x) + \frac{2}{q^{n-(n+p-1)}} \|f''\|_B$$

and the proof of the Lemma 4 is now completed.

**Theorem 1.** Let $(q_n) \subset (0,1)$ be a sequence such that $q_n \rightarrow 1$ as $n \rightarrow \infty$. Then for every $n > 2$, $x \in [0,\infty)$ and $f \in C_0[0,\infty)$, we have the inequality

$$|S_{n,p,q_n}(f,x) - f(x)| \leq C \omega_2\left( f, \sqrt{\delta_{n,p,q_n}(x)} \right) + \omega(f, \eta_{n,p,q_n}(x)).$$

where

$$\eta_{n,p,q_n}(x) = \frac{[n+p]_q}{q^{n-[n+p-1]l_n}} x + \frac{1}{q^{n-[n+p-1]l_n}}.$$

**Proof.** Using (13) for any $g \in W^2_{\infty}$, we obtain the inequality

$$|S_{n,p,q}(f,g,x) - f(x)| \leq |\tilde{S}_{n,p,q}(f-g,x) - (f-g)(x) + \tilde{S}_{n,p,q}(g,x) - g(x)|$$

$$+ |f\left( \frac{[n+p]_q}{q^{n-[n+p-1]l_n}} x + \frac{1}{q^{n-[n+p-1]l_n}} - f(x) \right)|.$$
From Lemma 3 and Lemma 4, we get
\[
|S_{n,p,q_0}(f) - f(x)| \leq 2\|f - g\|_B + \delta_{n,p,q_0}(x)\|g''\|_B \\
+ \left| \frac{f\left(\frac{[n+p]q_{0}}{q_{0}q_{0}'(n+p-1)} x + \frac{1}{q_{0}q_{0}'(n+p-1)}\right)}{q_{0}q_{0}'(n+p-1)} - f(x) \right|
\]
By using (11), we have
\[
|S_{n,p,q_0}(f) - f(x)| \leq 2\|f - g\|_B + \delta_{n,p,q_0}(x)\|g''\|_B + \omega(f, \eta_{n,p,q_0}(x)).
\]
Taking infimum over \( g \in W^2_B \) on the right hand side of the inequality (14) and using the inequality (12), we get the desired result.

4. WEIGHTED APPROXIMATION

The weighted Korovkin–type theorems was proved by Gadzhiev [19]. We give the Gadzhiev’s results in weighted spaces. Let \( p(x) = 1 + x^2, \) \( B_0[0, \infty) \) denotes the set of all functions \( f, \) from \([0, \infty)\) to \( R,\) satisfying the growth condition \( |f(x)| \leq \gamma f(x), \) where \( \gamma \) is a constant depending only on \( f. \) \( B_0[0, \infty) \) is a normed space with the norm \( \|f\|_\rho = \sup_{x \in [0, \infty)} \left|\frac{f(x)}{p(x)}\right| \) exists finitely.

**Theorem 2** Let \( (q_n) \subset (0,1) \) be a sequence such that \( q_n \to 1 \) as \( n \to \infty. \) Then for \( f \in C_0[0, \infty) \) and \( n > 1, \) we have \( \lim_{n \to \infty} \|S_{n,p,q_0}(f) - f\|_\rho = 0. \)

**Proof.** From Lemma 1, it is obvious that \( \|S_{n,p,q_0}(e_0) - e_0\|_\rho = 0. \) Let \( n > 2. \) Using Lemma we see that,
\[
|S_{n,p,q_0}(e_1, x) - e_1(x)| \leq \left| \frac{[n+p]q_{0}}{q_{0}q_{0}'(n+p-1)} x + \frac{1}{q_{0}q_{0}'(n+p-1)} - x \right|
\]
\[
= \left| \frac{[n+p]q_{0}}{q_{0}q_{0}'(n+p-1)} x + \frac{1}{q_{0}q_{0}'(n+p-1)} \right|
\]
\[
\leq \left( \frac{x}{q_{0}} + \frac{1}{q_{0}q_{0}'(n+p-1)} \right)
\]
\[
\leq \frac{2}{q_{0}} (x + 1)
\]
And we have
\[
\|S_{n,p,q_0}(e_1) - e_1\|_\rho \leq \sup_{x \in [0, \infty)} \frac{1 + x^2}{1 + x^2 q_{0}^2 (n + p - 1) q_{0}^2} (n + p - 1) q_{0}^2
\]
Therefore,
\[
\lim_{n \to \infty} \|S_{n,p,q_0}(e_1) - e_1\|_\rho = 0.
\]
Similarly,
\[
|S_{n,p,q_0}(e_2, x) - e_2(x)| \leq \left| \frac{[n+p]^2 q_{0}}{q_{0}q_{0}'(n+p-1)} \right| x^2 + \frac{[2] q_{0}}{q_{0}q_{0}'(n+p-1)} x + \frac{[2] q_{0}}{q_{0}q_{0}'(n+p-1)} x^2
\]
By the equality
\[
[n+p]^2 q_{0} - q_{0}^2 q_{0}'(n+p-1)
\]
\[
= \frac{1}{q_{0}^2 - q_{0}^2 q_{0}'(n+p-1)} (n + p - 1)
\]
\[
= \left( \frac{q_{0}^2 q_{0}'}{q_{0}^2 - q_{0}^2 q_{0}'} \right) \left( \frac{n + p - 1}{q_{0}^2 - q_{0}^2 q_{0}'} \right)
\]
We get following inequality
\[
|S_{n,p,q_0}(e_2, x) - e_2(x)| \leq (x^2 + x + 1) \frac{9(n + p + 4) q_{0}^2}{q_{0}^2 q_{0}'(n+p-1)}
\]
Hence,
\[ \|S_{n,p,q_n}(e_2) - e_2\| \leq \sup_{x \in [0,\infty)} 1 + x + x^2 \left( \frac{q[n+p+4]}{q^2F(n+p-1)} x + \frac{9}{128} x^3 + \frac{9}{128} x^2 + \frac{9}{128} x + 1 \right) \]

Then we have \( \lim_{n \to \infty} \|S_{n,p,q_n}(e_2) - e_2\| = 0 \).

Thus, from A. D. Gadzhiev’s Theorem in [25], we obtain desired result of Theorem 2.

5. RATE OF WEIGHTED APPROXIMATION

Now we want to estimate the rate of convergence for the sequence of the \( q \)–Szász-Schurer-beta operators \( S_{n,p,q} \). As it is known if \( f \) is not uniformly continuous on the interval \([0,\infty)\), then the usual first modulus of continuity \( w(f,\delta) \) does not tend to zero, as \( \delta \to 0 \). For every \( f \in C^0_0[0,\infty) \), we would like to take a weighted modulus of continuity \( \omega(f,\delta) \) which tends to zero as \( \delta \to 0 \). We consider the weighted modulus of continuity \( \omega(f,\delta) \) as

\[ \omega(f,\delta) = \sup_{0<h<\delta, x \in [0,\infty]} \frac{|f(x+h)-f(x)|}{(1+h^2)(1+x^2)}, \]

for each every \( f \in C^0_0[0,\infty) \).

(15)

The definition and properties of the weighted modulus of \( \omega(f,\delta) \) were given by Ispir in [26].

Now we will obtain the rate of convergence for the operators \( S_{n,p,q_n} \).

**Theorem 3.** Let \( f \in C^0_0[0,\infty) \) and \((q_n) \subset (0,1)\) be a sequence such that \( q_n \to 1 \) as \( n \to \infty \), then, we have the inequality

\[ \|S_{n,p,q_n}(f) - f\|_p \leq M(n,p,q_n) \omega \left( f, \frac{1}{\sqrt{n}} \left( 1 - q_n^4 + \frac{14+q_n^2}{F(n+p-1)} x + 1 \right) \right), \]

where \( \omega(x) = 1 + x^5 \) and \( M(n,p,q_n) \) is a positive real number dependent on \( n, p \) and \( q_n \) for \( n > 4 \).

**Proof.** From the definition of \( \omega(f,\delta) \) in (15), we get

\[ |f(t) - f(x)| \leq (1 + (t - x)^2)(1 + x^2) \left( 1 + \frac{|t-x|}{\delta} \right) \omega(f,\delta). \]

Then we yield the inequality

\[ |S_{n,p,q_n}(f(t),x) - f(x)| \leq \omega(f,\delta)(1 + x^2) S_{n,p,q_n} \left( (1 + (t - x)^2)(1 + \frac{|t-x|}{\delta}), x \right) \]

\[ \leq \omega(f,\delta)(1 + x^2) (S_{n,p,q_n}(1 + (t - x)^2, x) \]

\[ + S_{n,p,q_n} \left( (1 + (t - x)^2) \left( \frac{|t-x|}{\delta}, \right) \right). \]

(16)

Applying the Cauchy–Schwarz inequality in the last term of inequality (16) we obtain

\[ S_{n,p,q_n} \left( (1 + (t - x)^2) \left( \frac{|t-x|}{\delta}, \right) \right) \leq \left( S_{n,p,q_n}((1 + (t - x)^2, x) \right)^{\frac{1}{2}} \]

\[ \left( S_{n,p,q_n} \left( \left( \frac{|t-x|}{\delta}, \right) \right) \right)^{\frac{1}{2}}. \]

From Lemma 1 and Lemma 2, we have the following estimates.

\[ S_{n,p,q_n}((1 + (t - x)^2, x) \]

\[ = 1 + 2S_{n,p,q_n}((t - x)^2, x) + S_{n,p,q_n}((t - x)^4, x) \]

\[ \leq 2M_1(n,p,q_n)(x+1)^2 + \left( \frac{q_n^4 - q_n^2}{q_n^2F(n+p-1)} \right) \left( x^4 + x^3 + x^2 + x + 1 \right) \]

\[ \leq M_2(n,p,q_n)(x+1)^4, \]

and

using (17) and inequality (ii) of Lemma 2 we get

\[ S_{n,p,q_n}((1 + (t - x)^2, x) \]

\[ = 1 + 2S_{n,p,q_n}((t - x)^2, x) + S_{n,p,q_n}((t - x)^4, x) \]

\[ \leq 2M_1(n,p,q_n)(x+1)^2 + \left( \frac{q_n^4 - q_n^2}{q_n^2F(n+p-1)} \right) \left( x^4 + x^3 + x^2 + x + 1 \right) \]

\[ \leq M_2(n,p,q_n)(x+1)^4, \]

and
\[
\left( S_{n,p,q_n}, \left( \frac{(x-y)^2}{2}, x \right) \right)^{1/2} \leq \frac{1}{\Delta} \sqrt{\frac{2(1-q_n^2)}{q_n^2} \cdot \frac{2+2q_n^2}{q_n^{2n}(n+p+1)} (x+1)} \\
\leq M_{\alpha,n,p,q_n} \cdot \frac{1}{q_n^2} \sqrt{1 - q_n^2 \cdot \frac{1+q_n^2}{r_0(n+p-1)} (x+1)}
\]

Choosing \( M_{n,p,q_n} = \left( M_{1}(n,p,q_n) + \sqrt{M_{2}(n,p,q_n)} \right) M_{3}(n,p,q_n) M_4 \), where \( M_4 = \sup_{x \in disc} (1 + x^2) (1 + x^2)/(1 + x^3) \) and \( \Delta = \frac{1}{q_n^2} \sqrt{1 - q_n^2 + \frac{1+q_n^2}{r_0(n+p-1)} (x+1)} \) and using (17), (18), (19) in (16), we obtain

\[
|S_{n,p,q_n}(f,x) - f(x)| \leq (1 + x^2) M_{n,p,q_n} \left( \frac{1}{4\Delta} \sqrt{1 - q_n^2 + \frac{1+q_n^2}{r_0(n+p-1)}} \right) (x+1)
\]

Therefore the proof of the Theorem 3 is completed.

**CONFLICT OF INTEREST**

No conflict of interest was declared by the authors.

**REFERENCES**


