A General Fixed Point Theorem for Mappings Satisfying
An $\phi$ - Implicit Relation in Complete $G$ - Metric Spaces

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Received: 01.11.2011 Revised: 02.12.2011 Accepted: 23.12.2011

ABSTRACT
In this paper a general fixed point theorem for mappings satisfying an $\phi$ - implicit relation is proved, which generalize the results from [3] and [14].

Keywords: fixed point, $G$ - metric space, implicit relation, $\phi$ - implicit relation.

1. INTRODUCTION

In [4], [5] Dhage introduced a new class of generalized metric space, called $D$ - metric space. Mustafa and Sims [12], [13] proved that most of the claims concerning the fundamental topological structures of $D$ - metric spaces are incorrect and introduced an appropriate notion of generalized metric space, named $G$ - metric space. In fact, Mustafa and other authors [3], [7] – [15], [18] studied many fixed point results for self mappings in a $G$ - metric space under certain conditions. In [6] and [18] some fixed point theorems for mappings satisfying $\phi$ - maps are proved. In [16], [17], Popa initiated the study of fixed points for mappings satisfying implicit relations. In [2] Altun and Turkoglu introduced a new type of implicit relations satisfying a $\phi$ - map. The purpose of this paper is to prove a general fixed point theorem in $G$ - metric spaces for mappings satisfying an $\phi$ - implicit relation which generalize the results from [3] and [14].

2. PRELIMINARIES

Definition 2.1 ([13]) Let $X$ be a nonempty set and $G : X^3 \to \mathbb{R}_+$ satisfying the following properties:

$(G_1) : G(x,y,z) = 0$ if $x = y = z$ ;
$(G_2) : 0 < G(x,y,z) \leq G(x,x,y)$ for all $x, y \in X$ with $x \neq y$ ;
$(G_3) : G(x,y,z) \leq G(x,y,z)$ for all $x, y, z \in X$ with $y \neq z$ ;
$(G_4) : G(x,y,z) = G(x,z,y) = G(y,z,x) =$...

(symmetry in all three variables);
$(G_5) : G(x,y,z) \leq G(x,a,a) + G(a,y,z)$ for all $x, y, z, a \in X$ (rectangle inequality).

Then, the function $G$ is called a $G$ - metric on $X$ and $(X,G)$ is called a $G$ - metric space.

Note that if $G(x,y,z) = 0$ then $x = y = z$ .

Definition 2.2 ([13]) Let $(X, G)$ be a $G$ - metric space. A sequence $(x_n)$ in $X$ is said to be
\* \* G - convergent if for \( \varepsilon > 0 \), there exists an \( x \in X \) and \( k \in \mathbb{N} \) such that for all \( m, n \geq k \), 
\[ G(x, x_m, x_n) < \varepsilon . \]

\* \* G - Cauchy if for each \( \varepsilon > 0 \), there exists \( k \in \mathbb{N} \) such that for all \( m, n, p \geq k \), 
\[ G(x_m, x_n, x_p) < \varepsilon , \] that is 
\[ G(x_m, x_n, x_p) \to 0 \text{ as } n, m, p \to \infty . \]

A space \((X, G)\) is called \( G \)-complete if every \( G \)-Cauchy sequence in \((X, G)\) is \( G \)-convergent.

**Lemma 2.1** ([13]) Let \((X, G)\) be a \( G \)-metric space. Then the following properties are equivalent:
1) \((x_n)\) is \( G \)-convergent to \( x \);
2) \( G(x_n, x, x) \to 0 \text{ as } n \to \infty ; \)
3) \( G(x_n, x, x) \to 0 \text{ as } n \to \infty ; \)
4) \( G(x_m, x_n, x) \to 0 \text{ as } m, n \to \infty . \)

**Lemma 2.2** ([13]) If \((X, G)\) is a \( G \)-metric space, then the following are equivalent:
1) The sequence \((x_n)\) is \( G \)-Cauchy;
2) For every \( \varepsilon > 0 \), there exists \( k \in \mathbb{N} \) such that \( G(x_n, x_m, x_m) < \varepsilon \) for \( n, m > k \).

**Lemma 2.3** ([13]) Let \((X, G)\) be a \( G \)-metric space. Then, the function \( G(x, y, z) \) is jointly continuous in all three of its variables.

### 3. IMPLICIT RELATIONS

**Definition 3.1.** A function \( f : [0, \infty) \to [0, \infty) \) is a \( \phi \)-function, \( f \in \Phi \) if \( f \) is a nondecreasing function such that 
\[ \sum_{k=1}^{\infty} f^{(k)}(t) < \infty , \] for all \( f(t) < t \) for \( t > 0 \) and \( f(0) = 0 . \)

**Definition 3.2.** Let \( F_0 \) be the set of all continuous functions \( F(t_1, \ldots, t_6) : \mathbb{R}_+^6 \to \mathbb{R} \) such that:
1) \( F \) is nonincreasing in \( t_5 \),
2) There exists a function \( \phi_1 \in \Phi \) such that for all \( u, v \geq 0 \), \( F(u, v, u, u + v, 0) \leq 0 \) implies \( u \leq \phi_1(v) \).
3) There exists a function \( \phi_2 \in \Phi \) such that for all \( t, t' > 0 \), 
\[ F(t, t, 0, t, t') \leq 0 \] implies \( t \leq \phi_2(t') \).

**Example 3.1.**
\[
F(t_1, \ldots, t_6) = t_1 - t_2 - bt_3 - ct_4 - dt_5 - et_6 , \quad \text{where} \quad a > 0 , \ b, c, d, e \geq 0 , \ a + b + c + 2d + e \leq 1 .
\]

1) \( F_1 \) : Obviously.
2) \( F_2 \) : Let \( u, v \geq 0 \) be and 
\[ F(u, v, u, u + v, 0) = u - av - bv - cu - d(u + v) \leq 0 \] 
which implies \( u \leq \frac{a + b + d}{1 - c - d} v \) and \( F_2 \) is satisfied for
\[ \phi_1(t) = \frac{a + b + d}{1 - (a + d)} t . \]
\( F_2 \) : Let \( t, t' > 0 \) be and
\[ F(t, t, 0, 0, t, t') = t - at - dt - et' \leq 0 \] which implies 
\[ t \leq \frac{e}{1 - (a + d)} t' \] and \( F_2 \) is satisfied for
\[ \phi_2(t) = \frac{e}{1 - (a + d)} t . \]

**Example 3.2.**
\[ F(t_1, \ldots, t_6) = t_1 - k \max\{t_2, \ldots, t_6\} , \quad \text{where} \quad k \in \left\{ \frac{1}{2} \right\} . \]

1) \( F_1 \) : Obviously.
2) \( F_2 \) : Let \( u, v \geq 0 \) be and 
\[ F(u, v, u, u + v, 0) = u - k \max\{u, v, u + v\} \leq 0 . \] Hence 
\[ u \leq \frac{k}{1 - k} v \] and \( F_2 \) is satisfied for \( \phi_1(t) = \frac{k}{1 - k} t . \)

3) \( F_3 \) : Let \( t, t' > 0 \) be and
\[ F(t, t, 0, 0, t, t') = t - k \max\{t, t'\} \leq 0 . \] If \( t > t' \), then 
\[ t(1 - k) \leq 0 , \] a contradiction. Hence \( t \leq t' \) which implies \( t \leq k t' \) and \( F_3 \) is satisfied for \( \phi_2(t) = kt . \)

**Example 3.3.**
\[ F(t_1, \ldots, t_6) = t_1 - k \max\{t_2, t_3, t_4, t_5 + t_6\} , \quad \text{where} \quad k \in (0, 1) . \]

1) \( F_1 \) : Obviously.
2) \( F_2 \) : Let \( u, v \geq 0 \) be and 
\[ F(u, v, u, u + v, 0) = u - k \max\{u, v, u + v\} \leq 0 . \] If 
\[ u > v , \] then \( u(1 - k) \leq 0 , \] a contradiction. Hence \( u \leq v \) which implies \( u \leq kv \) and \( F_2 \) is satisfied for \( \phi_1(t) = kt . \)
3) \( F_3 \) : Let \( t, t' > 0 \) be and
\[ F(t, t, 0, 0, t, t') = t - k \max\{t, t'\} \leq 0 . \] If \( t > t' \), then 
\[ t(1 - k) \leq 0 , \] a contradiction. Hence \( t \leq t' \) which implies \( t \leq k t' \) and \( F_3 \) is satisfied for \( \phi_2(t) = kt . \)

**Example 3.4.**
\[ F(t_1, \ldots, t_6) = t_1^2 - t_1(at_2 + bt_3 + ct_4) - dt_5t_6 , \quad \text{where} \quad a > 0 , \ b, c, d \geq 0 , \ a + b + c < 1 . \]

1) \( F_1 \) : Obviously.
2) \( F_2 \) : Let \( u, v \geq 0 \) be and 
\[ F(u, v, u, u + v, 0) = u^2 - u(av + bv + cu) \leq 0 . \] If 
\[ u > 0 , \] then 
\[ u \leq \frac{1 - c}{1 - c} v . \] If \( u = 0 \), then 
\[ u \leq \frac{1 - c}{1 - c} v \] and
\( F_2 \) is satisfied for \( \phi_1(t) = \frac{a + b}{1 - c} t . \)
Example 3.7. \[ F(t, t, 0, 0, t') = t^2 - at^2 - ct' \leq 0 \], which implies \( t \leq \frac{c}{1-a} t' \) and \((F_3)\) is satisfied for \( \phi_2(t) = \frac{c}{1-a} t \).

Example 3.8. \[ F(t_1, ..., t_6) = t_1 - at_2 - bt_3 - c \max\{2t_4, t_5 + t_6\} \]
where \( a > 0 \), \( b, c \geq 0 \), \( a + b + 2c < 1 \).

\((F_1)\): Obviously.
\((F_2)\): Let \( u, v \geq 0 \) be and \( F(u, v, u, u + v, 0) = u - av - bv - c \max\{2u, u + v\} \leq 0 \).
If \( u > v \), then \( u(1 - k) \leq 0 \), a contradiction. Hence \( u \leq v \) which implies \( u \leq kv \) and \((F_2)\) is satisfied for \( \phi_1(t) = kt \).
\(F_3)\): Let \( t, t' > 0 \) be and \( F(t, t, 0, 0, t') = t - at - c(t + t') \leq 0 \), which implies \( t \leq \frac{c}{1-a} t' \) and \((F_3)\) is satisfied for \( \phi_2(t) = \frac{c}{1-a} t \).

Example 3.9. \[ F(t_1, ..., t_6) = t_1 - at_2 - bt_3 - c \max\{t_4 + t_5, 2t_6\} \]
where \( a > 0 \), \( b, c \geq 0 \), \( a + b + 3c < 1 \).

\((F_1)\): Obviously.
\((F_2)\): Let \( u, v \geq 0 \) be and \( F(u, v, v, u, u + v, 0) = u - av - bv - c(2u + v) \leq 0 \)
which implies \( u \leq \frac{a + b + c}{1 - 2c} v \) and \((F_2)\) is satisfied for \( \phi_1(t) = \frac{a + b + c}{1 - 2c} t \).
\((F_3)\): Let \( t, t' > 0 \) be and \( F(t, t, 0, 0, t', t') = t - at - c(t + t') \leq 0 \), which implies \( t \leq \frac{c}{1-a} t' \) and \((F_3)\) is satisfied for \( \phi_2(t) = \frac{c}{1-a} t \).

Example 3.10. \[ F(t_1, ..., t_6) = t_1 - c \max\{t_2, t_3, 2t_4, t_5 + t_6\} \]
where \( a > 0 \), \( c \geq 0 \), \( a + c < 1 \).

\((F_1)\): Obviously.
\((F_2)\): Let \( u, v \geq 0 \) be and \( F(u, v, v, u, u + v, 0) = u - cv \leq 0 \) which implies \( u \leq cv \) and \((F_2)\) is satisfied for \( \phi_1(t) = ct \).
\((F_3)\): Let \( t, t' > 0 \) be and \( F(t, t, 0, 0, t', t') = t - c \max\{t, t'\} \leq 0 \).
If \( t > t' \) then \( t(1 - c) \leq 0 \), a contradiction. Hence \( t \leq t' \) which
implies \( t \leq ct' \) and \((P_2)\) is satisfied for \( \phi_2(t) = ct \).

4. MAIN RESULTS

**Theorem 4.1** Let \((X, G)\) be a \(G\) - metric space. Suppose that
\[
F(G(Tx,Ty,Tz), G(x,y), G(x,Tx,Tx), G(y,Ty,Ty), G(y,Tx,Tx)) \leq 0, \tag{4.1}
\]
for all \(x, y \in X\) where \(F\) satisfies condition \((F_2)\). Then \(T\) has at most a fixed point.

**Proof.** Suppose that \(T\) has two distinct fixed points \(u, v\). Then by \((4.1)\) we have successively
\[
F(G(u,v,v), G(u,v), v, 0, 0, G(u,v,v), G(v,u,u)) \leq 0,
\]
which implies by \((F_2)\) that
\[
G(u,v,v) \leq \phi_2(G(v,u,u)).
\]
Similarly, we have
\[
G(v,u,v) \leq \phi_2(G(u,v,v)).
\]
Hence
\[
G(u,v,v) \leq \phi_2(G(v,u,u)) \leq \phi_2^2(G(u,v,v)) < G(u,v,v),
\]
a contradiction. Hence \(u = v\).

**Theorem 4.2** Let \((X, G)\) be a complete \(G\) - metric space. Suppose that \((4.1)\) holds for all \(x, y \in X\) and \(F \in F_\phi\). Then \(T\) has a unique fixed point.

**Proof.** By \((4.1)\) for \(y = Tx\) we obtain
\[
F(G(Tx,T^2x,T^2x), G(x,Tx,Tx), G(x,Tx,Tx), G(x,T^2x,T^2x), 0) \leq 0.
\]
By rectangle inequality and \((F_1)\) we have that
\[
F(G(Tx,T^2x,T^2x), G(x,Tx,Tx), G(x,Tx,Tx)), G(Tx,T^2x,T^2x), 0) \leq 0.
\]
By \((F_2)\) we obtain
\[
G(Tx,T^2x,T^2x) \leq \phi_1(G(x,Tx,Tx)). \tag{4.2}
\]
Let \(x_0 \in X\) be and \(x_n = Tx_{n-1}, n = 0,1,2,\ldots\) Hence
\[
G(x_n,x_{n+1}) \leq \phi_1^n(G(x_{n-1},x_n)) \leq \phi_1^{m-1}(G(x_0,x_1)).
\]
By rectangle inequality we obtain
\[
G(x_n,x_{n+k}) \leq \phi_1^n(G(x_{n+k},x_{n+k+1},x_{n+k+2},\ldots)) + \phi_1^n(G(x_{n+k+1},x_{n+k+2})) + \ldots + \phi_1^n(G(x_{n+k},x_{n+k+1})) \leq \phi_1^n(G(x_0,x_1)) \leq \sum_{k=0}^{m-1} \phi_1^n(G(x_0,x_1)).
\]
Let \(\varepsilon > 0\). Since \(\sum_{k=1}^\infty \phi_1^n(G(x_0,x_1)) < \infty\), there exists \(k \in \mathbb{N}\) such that for \(m > n \geq k\),
\[
\sum_{k=n+1}^{m-1} \phi_1^n(G(x_0,x_1)) < \varepsilon.
\]
It follows by Lemma 2.2 that \(\{x_n\}\) is a \(G\) - Cauchy sequence in a complete \(G\) - metric space and so has a limit \(u\). We prove that \(u = Tu\) by \((4.1)\) we have successively
\[
F(G(Tx,Tu,Tu), G(x_n,u,u), G(x_n,Tx,Tx), G(u,Tu,Tu), G(x_n,Tu,Tu), G(u,Tx,Tx)) \leq 0.
\]
Hence by \((F_2)\) we obtain
\[
G(u,u,u) \leq \phi_1(G(u,u,u)) \leq \phi_1^2(G(u,u,u)) < G(u,u,u),
\]
which implies that \(u = Tu\), \(u\) is a fixed point of \(T\). By (4.1) \((4.1)\) is the unique fixed point of \(T\).

**Corollary 4.1 (Theorem 2.1 [14])** Let \((X, G)\) be a complete \(G\) - metric space and let \(T : X \to X\) be a mapping satisfying the following inequality for all \(x, y, z \in X\)
\[
G(Tx,Ty,Tz) \leq k \max\{G(x,y,z), G(x,Tx,Tx), G(y,Ty,Ty), G(z,Tz,Tz), G(x,Tz,Tz), G(z,Tx,Tx)\}, \tag{4.3}
\]
where \(k \in \left[0, \frac{1}{2}\right]\). Then, \(T\) has a unique fixed point.

**Proof.** If \(z = y\) by \((4.3)\) we obtain
\[
G(Tx,Ty,Ty) \leq k \max\{G(x,y), G(x,Tx), G(y,Ty), G(x,Ty), G(y,Ty), 0, G(y,Tx), G(y,Tx)\}.
\]
By Theorem 4.2 and Example 3.2, \(T\) has a unique fixed point.

**Corollary 4.2 (Theorem 3.2 [14])** Let \((X, G)\) be a complete \(G\) - metric space and let \(T : X \to X\) be a mapping satisfying the following inequality for all \(x, y, z \in X\)
\[
G(Tx,Ty,Tz) \leq k \max\{G(x,Ty,Ty) + G(y,Tx,Tx), G(y,Tz,Tz) + G(z,Ty,Ty)\}, \tag{4.4}
\]
where \(k \in (0,1)\). Then, \(T\) has a unique fixed point.

**Proof.** If \(z = y\) we obtain by \((4.4)\) that
\[
G(Tx,Ty,Ty) \leq k \max\{G(x,Ty,Ty) + G(y,Tx,Tx), 2G(y,Ty,Ty)\}.
\]
By Theorem 4.2 and Example 3.9 with \(a = b = 0\) and \(c = k\), then \(T\) has a unique fixed point.

**Corollary 4.3 (Theorem 2.6 [14])** Let \((X, G)\) be a complete \(G\) - metric space and let \(T : X \to X\) be a mapping satisfying the following inequality for all \(x, y, z \in X\)
\[
G(Tx,Ty,Tz) \leq k \max\{G(x,Ty,Ty) + G(x,Ty,Ty), 2G(y,Ty,Ty)\}, \tag{4.5}
\]
where \(k \in (0,1)\). Then, \(T\) has a unique fixed point.
where $k \in \left(0, \frac{1}{2}\right)$. Then, $T$ has a unique fixed point.

Proof. By Theorem 4.2 and Example 3.9 with $a = b = 0$ and $c = k$, then $T$ has a unique fixed point.

**Corollary 4.4 (Theorem 2.8 [14])** Let $(X, G)$ be a complete $G$- metric space and let $T : X \to X$ be a mapping satisfying the following inequality for all $x, y, z \in X$

$$G(Tx, Ty, Tz) \leq k \max\{G(z, Tx, Tx) + G(y, Tx, Tx), G(y, Tz, Tz), G(x, Tx, Tx)\} + G(y, Ty, Ty)$$

where $k \in \left(0, \frac{1}{2}\right)$. Then, $T$ has a unique fixed point.

Proof. If $z = y$ by (4.6) we obtain

$$G(Tx, Ty, Ty) \leq k \max\{G(y, y, y), G(x, Tx, Tx)\} + G(x, Ty, Ty), 2G(y, Tx, Tx)$$

and the proof follows by Corollary 4.3.

**Corollary 4.5 (Theorem 2.1 [3])** Let $(X, G)$ be a complete $G$- metric space and let $T : X \to X$ be a mapping satisfying the following inequality for all $x, y, z \in X$

$$G(Tx, Ty, Tz) \leq k \max\{G(x, y, z), G(x, Tx, Tx), G(y, Ty, Ty), \frac{G(x, Ty, Ty) + G(z, Tx, Tx)}{2}, \frac{G(y, Tz, Tz) + G(z, Tx, Tx)}{2}, \frac{G(y, Ty, Ty)}{2}\}$$

where $k \in (0,1)$. Then, $T$ has a unique fixed point.

Proof. If $z = y$ by (4.7) we obtain

$$G(Tx, Ty, Ty) \leq k \max\{G(x, y, y), G(x, Tx, Tx), G(y, Ty, Ty), \frac{G(x, Ty, Ty) + G(y, Tx, Tx)}{2}\}.$$

By Theorem 4.2 and Example 3.3, $T$ has a unique fixed point.

**Corollary 4.6 (Theorem 2.2 [3])** Let $(X, G)$ be a complete $G$- metric space and let $T : X \to X$ be a mapping satisfying the following inequality for all $x, y, z \in X$

$$G(Tx, Ty, Tz) \leq k \max\{G(x, y, z), G(x, Tx, Tx), G(y, Ty, Ty), G(x, Ty, Ty), G(z, Tx, Tx)\}$$

where $k \in \left(0, \frac{1}{2}\right)$. Then, $T$ has a unique fixed point.

Proof. If $y = z$ we obtain

$$G(Tx, Ty, Ty) \leq k \max\{G(x, y, y), G(x, Tx, Tx), G(y, Ty, Ty), G(x, Ty, Ty), G(y, Tx, Tx)\}.$$


