Coincidence Points of Hybrid Functions on Cone Metric Spaces

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ABSTRACT

In this paper, we obtain a common coincidence point theorem for two pairs of hybrid functions on cone metric spaces.

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1. INTRODUCTION

In 2007, Huang and Zhang defined cone metric spaces by substituting an ordered normed space for the real numbers\(^{[9]}\). In 2008, Rezapour and Hambarani characterized types of cones\(^{[17]}\). Some interesting works about fixed point and common fixed point results on cone metric spaces are\(^{[1-8,10,11,13-24]}\) etc.

In this paper, we prove a common coincidence point theorem for two pairs of hybrid functions on cone metric spaces. Our result generalizes and improves the theorems of\(^{[18,19]}\). First we give some known definitions and lemmas.

Let \(E\) be a real Banach space and \(P\) a subset of \(E\). \(P\) is called a cone whenever:

(i) \(P\) is closed, non empty and \(P\) is a cone whenever

(ii) \(ax + by\) for all \(x, y \in P\) and non negative real numbers \(a\) and \(b\)

(iii) \(P \cap (-P) = \{0\}\).

For a given cone \(P \subseteq E\), we can define a partial ordering \(\leq\) with respect to \(P\) by \(x \leq y\) if and only if \(y - x \in P\). \(x < y\) will stand for \(x \leq y\) and \(x \not\leq y\), while \(x << y\) will stand for \(y - x \in \text{int } P\), where \(\text{int } P\) denotes the interior of \(P\).

The cone \(P\) is called normal if there is a number \(M > 0\) such that for all \(x, y \in E\), \(0 \leq x \leq y\) implies

\[ \|x\| \leq M \|y\|. \]

The least positive number satisfying the above inequality is called the normal constant of \(P\).

Rezapour and Hambarani\(^{[17]}\) observed that there are no normal cones with \(M < 1\). Hence \(M \geq 1\).

Definition 1.1.\(^{[9]}\): Let \(X\) be a nonempty set. Suppose that the mapping \(d : X \times X \to E\) satisfies

\(d_1\) for all \(x, y \in X\) and \(d(x, y) = 0\) if and only if \(x = y\),

\(d_2\) for all \(x, y, z \in X\).

Then \(d\) is called a cone metric on \(X\) and \((X, d)\) is called a cone metric space.

Definition 1.2.\(^{[9]}\): Let \((X, d)\) be a cone metric space, \(x \in X\) and \(\{x_n\}\) a sequence in \(X\). Then

(i) \(\{x_n\}\) converges to \(x\) whenever for every \(c \in E\) with \(0 \leq c\), there is a natural
number $N$ such that $d(x_n, x) << c$ for all $n \geq N$, we denote this by $\lim_{n \to \infty} x_n = x$ or $x_n \to x$.

(ii) \{ $x_n$ \} is a Cauchy sequence whenever for every $c \in E$ with $0 << c$, there is a natural number $N$ such that $d(x_n, x_m) << c$ for all $n, m \geq N$.

(iii) $(X, d)$ is a complete metric space if every Cauchy sequence in $X$ is convergent in $X$.

Definition 1.3. [18]: Let $(X, d)$ be a cone metric space and $B \subseteq X$.

(i) A point $b \in B$ is called an interior point of $B$ whenever there is a $0 << p$ such that $N(b, p) = \{y \in X : d(y, b) << p \}$.

(ii) A subset $A \subseteq X$ is called open if each element of $A$ is an interior point of $A$.

The family $\beta = \{N(x, p) : x \in X, 0 << p \}$ is a sub basis for a topology on $X$.

we denote this cone topology by $\tau_c$. Then $\tau_c$ is Hausdorff and first countable.

Recently Rezapour and Haghi [18] proved the following

Lemma 1.4. (Lemma 2.1, [18]): Let $(X, d)$ be a cone metric space, $P$ a normal cone with normal constant $M = 1$ and $A$ a compact set in $(X, \tau_c)$. Then for every $x \in X$, there exists $a_0 \in A$ such that

$$\inf_{a \in A} d(x, a) =: d(x, a_0).$$

Lemma 1.5. [Lemma 2.2, [19]]: Let $(X, d)$ be a cone metric space, $P$ a normal cone with normal constant $M = 1$ and $A, B$ two compact sets in $(X, \tau_c)$. Then

$$\sup_{x \in X} d^1(x, A) < \infty,$$

where $d^1(x, A) = \inf_{a \in A} d(x, a)$.

Definition 1.6. [18]: Let $(X, d)$ be a cone metric space, $P$ a normal cone with normal constant $M = 1$, $\mathcal{F}(X)$ be the set of all compact subsets of $(X, \tau_c)$ and $A \subseteq \mathcal{F}(X)$. Define $h_A : \mathcal{F}(X) \to [0, \infty)$ and

$$d_H : \mathcal{F}(X) \times \mathcal{F}(X) \to [0, \infty)$$

by $h_B(A) = \sup_{x \in A} d^1(x, B)$ and $d_H(A, B) = \max\{h_A(B), h_B(A)\}$ respectively. For each $A, B \subseteq \mathcal{F}(X)$ and $x, y \in A$, we have

(i) $d^1(x, A) \leq d(x, y) + d^1(y, A)$

(ii) $d^1(x, A) \leq d^1(x, B) + h_B(A)$

(iii) $d^1(x, A) \leq \sup_{y \in Y} d^1(x, y) + d^1(y, B) + h_B(A)$.

Definition 1.7. : Let $f : X \to X$ and $F : X \to \mathcal{F}(X)$. $f$ is said to be $F$-weakly commuting at $x \in X$ if $f^2 x \in F f x$.

Kamran [12] defined the above in metric spaces.

Definition 1.8. : Let $\phi$ denote the class of all functions $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\phi$ is non decreasing, continuous and $\sum_{n=1}^{\infty} (\phi^n)(t) < \infty$ for all $t > 0$.

It is clear that $\phi^n(t) \to 0$ as $n \to \infty$ for all $t > 0$ and hence, we have $\phi(t) < t$, for all $t > 0$.

Now we give our main result.

2. THE MAIN RESULT

Theorem 2.1. Let $(X, d)$ be a complete cone metric space with normal constant $M = 1$. Let $F, G : X \to \mathcal{F}(X)$ be two multifunctions and $f, g : X \to X$ be self maps satisfying

$$(2.1.1) d_H(Fx, Gy) \leq \phi,$$

for all $x, y \in X$ and $\phi \in \phi$.

(2.1.2) $F x \subseteq g(X), G(x) \subseteq f(X)$ for all $x \in X$.

(2.1.3) one of $f(X)$ and $g(X)$ is a complete subset of $X$ and

(2.1.4) $f$ is $F$-weakly commuting and $g$ is $G$-weakly commuting at their coincidence points.

Then the pairs $(f, F)$ and $(g, G)$ have the same coincidence point in $X$.

Proof. Let $x_0 \in X$. Then by Lemma 1.4, there exists $x_\infty \in F x_0$ such that

$$d^1(fx_0, Fx_0) = \sup_{x \in X}(d^1(x_0, A) + d^1(fx_0, A)).$$

Again by Lemma 1.4, there exists $x_\infty \in G x_1$ such that

$$d^1(gx_1, Gx_1) = \sup_{x \in X}(d^1(x_\infty, A) + d^1(gx_1, A)).$$

Continuing in this way, we get the sequences \{ $x_n$ \} and \{ $y_n$ \} in $X$ such that

$$d^1(y_{2n+1}, Fx_{2n+1}) = d^1(y_{2n+1}, Gx_{2n+1})$$

$$= \sup_{x \in X}(d^1(y_{2n+1}, A) + d^1(fx_{2n+1}, A) + d^1(gx_{2n+1}, A)).$$

Case(i): Suppose $y_{2n+1} \neq y_{2n+2}$.

Assume that $y_{2n+1} \neq y_{2n+2}$.

$$d^1(y_{2n+1}, Gx_{2n+1}) \leq d_H(F x_{2n+2}, G x_{2n+1}).$$
\[ \begin{align*}
\|d(y_{2n+1}, y_{2n+2})\| & \leq \phi \left( \sum_{i=n+1}^{\infty} \|d(y_i, y_{i+1})\| \right) \\
& \leq \phi (\|d(y_0, y_1)\| + \phi^{n+1} (\|d(y_n, y_{n+1})\| + \phi^n (\|d(y_0, y_1)\|) \\
& \leq \sum_{i=1}^{n} \phi^i (\|d(y_i, y_{i+1})\| \\
& \rightarrow 0 \text{ as } m \rightarrow \infty, \text{ since } \sum_{n=1}^{\infty} \phi^n (1) < \infty \\
\end{align*} \]

This is implies that \( \lim_{m,n \rightarrow \infty} \|d(y_n, y_m)\| = 0 \).

By Lemma 4.1, \( \{y_n\} \) is a Cauchy sequence in \( X \).

Suppose \( g(X) \) is complete.

Then \( y_{2n} = g x_{2n-1} \rightarrow p = g v \in g(X) \) for some \( p \) and \( v \in X \).

Since \( \{y_n\} \) is Cauchy, we have \( y_{2n+1} \rightarrow p \).

\[ \begin{align*}
\|d'(p, G v)\| & \leq \|d(p, y_{2n})\| + d'(y_{2n}, G v) \\
& \leq \|d(p, y_{2n})\| + d_{H}(F x_{2n}, G v) \\
& \leq \|d(p, y_{2n})\| + \phi \\
& \leq \|d'(y_{2n}, G v)\| + \|d'(y_{2n}, F x_{2n})\| \\
& \leq \|d'(y_{2n}, G v)\| + \phi \\
& \leq \|d'(y_{2n}, G v)\| + d'(p, G v) + \|d(p, y_{2n})\| \\
& \leq \|d'(y_{2n}, G v)\| + d'(p, G v) + \|d(p, y_{2n})\| \\
\end{align*} \]

Letting \( n \rightarrow \infty \), we get
\[ d'(p, G v) \leq \phi (d'(p, G v)) \text{ so that } d'(p, G v) = 0. \]

Hence \( p \in G v \). Thus \( g v = p \in G v \).

Since \( g \) is \( G \)-weakly commuting at the coincidence point \( v \), we have \( g p = g^2 v \in G g v = G p \). Thus \( p \) is a coincidence point of \( g \) and \( G \). Since \( G v \subseteq f(X) \), there exists \( w \in X \) such that \( p = g v = f w \in G v \).

\[ \begin{align*}
d'(p, F w) & \leq \|d(p, y_{2n+1})\| + d'(y_{2n+1}, F w) \\
& \leq \|d(p, y_{2n+1})\| + d_{H}(F w, G x_{2n+1}) \\
& \leq \|d(p, y_{2n+1})\| + \phi \\
\end{align*} \]
\[
\left\{ \begin{array}{l}
\max \left\{ \frac{1}{2}d(fw, Gx \\& y) + d'(y, Gw) \right\} \\
\max \left\{ \frac{1}{2}d(\phi(w, Gx \\& y)) + d'(y, Gw) \right\}
\end{array} \right\}
\leq \phi \left( d(\phi(w, Gx \\& y)) + d'(y, Gw) \right)
\]

Letting \( n \to \infty \), we get
\[ d_1(p, Fw) \leq \phi \left( d_1(p, Fw) \right) \text{ so that } d_1(p, Fw) = 0. \]

Hence \( p \in Fw \). Thus \( f w = p \in Fw \).

Since \( f \) is \( F \)-weakly commuting at the coincidence point \( w \), we have \( Fp = f Fw \in Fw = Fp \).

Thus \( p \) is a coincidence point of \( f \) and \( F \). Hence, the \( (p, F) \) and \( (g, G) \) have the same coincidence point.

By putting \( f = g = I \) (the identity map) in Theorem 2.1, we have

**Corollary 2.2.** Let \( (X, d) \) be a complete cone metric space with normal constant \( M = 1 \). Let \( F, G : X \to \mathcal{F} \) be two multi functions satisfying

\[
(2.2.1) d(H(Fx, Gx)) \leq \alpha \left\{ \begin{array}{l}
d(x, y) + d'(x, Fx) + d'(y, Gx) \\
\max \left\{ \frac{1}{2}d(x, y) + d'(x, Gx) + d'(y, Fx) \right\}
\end{array} \right\}
\]

for all \( x, y \in X \), where \( \alpha \in [0, 1) \).

Then \( F \) and \( G \) have a common fixed point in \( X \).

Corollary 2.2 is a generalization and improvement of Theorems 1 and 2 of [19] for a pair of multi functions and of Theorems 2.6 and 2.7 of [18] for a single multi function with \( F = G \).

**REFERENCES**


