Chebyshev Polynomial Solutions of Certain Second Order Non-Linear Differential Equations

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ABSTRACT

The purpose of this study is to give a Chebyshev polynomial approximation for the solution of second-order non-linear differential equations with variable coefficients. For this purpose, Chebyshev matrix method is introduced. This method is based on taking the truncated Chebyshev expansions of the functions in the non-linear differential equations. Hence, the result matrix equation can be solved and the unknown Chebyshev coefficients can be found approximately. Additionally, the mentioned method is illustrated by two examples.

Key Words: Non-linear differential equations, Chebyshev-matrix method, Approximate solution of non-linear ordinary differential equations.

1. INTRODUCTION

The Chebyshev matrix method has been presented by Keşan [2] to solve linear differential equations. This method has been also used by Köroğlu [3] to solve linear Fredholm integrodifferential equations. Additionally, Cantürk-Günhan [1] has extended this method to solve non-linear differential and integral equations. In this work, we adapt Chebyshev -Matrix method to second-order non-linear differential equations. It is presented as follow:

\[
\sum_{k=1}^{s} P_k(x)(y^n)^k + \sum_{k=1}^{m} Q_k(x)(y')^k + \sum_{k=1}^{n} R_k(x)(y)^k = F(x) \quad s, m, n = 1, 2, ...
\]  

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where \( P_k(x) \), \( Q_k(x) \), \( R_k(x) \) and \( F(x) \) are functions having Chebyshev expansions on an interval \( a \leq x \leq b \), under the given conditions, which are

\[
\sum_{i=0}^{2} [a_i y^{(i)}(a) + b_i y^{(i)}(b) + c_i y^{(i)}(c)] = \lambda \tag{2}
\]

\[
\sum_{i=0}^{2} [a_i y^{(i)}(a) + b_i y^{(i)}(b) + c_i y^{(i)}(c)] = \mu
\]

where \( a \leq c \leq b \), provide that the real coefficients \( a_i, b_i, c_i, \lambda \) and \( \mu \) are appropriate constants; and the solution is expressed in the form

\[
y(x) = \sum_{r=0}^{n} a_r T_r(x) \tag{3}
\]

under the certain conditions \( r = 0,1,2,\ldots \). And, \( \sum' \) denotes a sum whose first term is halved.

2. FUNDAMENTAL RELATIONS

Let us assume that the function \( y(x) \) its \( n \)th derivative with respect to \( x \), respectively, can be expanded in Chebyshev series

\[
y(x) = \sum_{r=0}^{\infty} a_r T_r(x) \quad \text{and} \quad y^{(n)}(x) = \sum_{r=0}^{\infty} a^{(n)}_r T_r(x).
\]

The recurrence relation between the Chebyshev coefficients \( a_r \) and \( a^{(n)}_r \) of \( y^{(n)}(x) \) and \( y^{(n+1)}(x) \), is given by

\[
a^{(n+1)}_r = 2 \sum_{i=0}^{\infty} (r + 2i + 1)a^{(n)}_{r+2i+1} \tag{4}
\]

Now let us take \( r=0,1,\ldots,N \) and assume \( a^{(n)}_r = 0 \) for \( r > N \). Then the system (4) can be transformed into the matrix form

\[
A^{(n+1)} = 2MA^{(n)} , \quad n = 0,1,2,\ldots \tag{5}
\]

where

\[
A^{(n)} = \left[ \frac{1}{2}a^{(n)}_0 \quad a^{(n)}_1 \ldots \quad a^{(n)}_N \right]^T
\]

for odd \( N \)

\[
M = \begin{bmatrix}
0 & 1 & 0 & 3 & 0 & 5 & \ldots & N \\
0 & 2 & 0 & 4 & 0 & \ldots & 0 \\
0 & 0 & 0 & 3 & 0 & 5 & \ldots & N \\
& \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
0 & 0 & 0 & 0 & 0 & 0 & \ldots & 0 \\
\end{bmatrix}_{(N+1) \times (N+1)}
\]

and for even \( N \)

\[
M = \begin{bmatrix}
0 & 1 & 0 & 3 & 0 & 5 & \ldots & 0 \\
0 & 2 & 0 & 4 & 0 & \ldots & N \\
0 & 0 & 0 & 3 & 0 & 5 & \ldots & 0 \\
& \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
0 & 0 & 0 & 0 & 0 & 0 & \ldots & 0 \\
\end{bmatrix}_{(N+1) \times (N+1)}
\]

It follows from relation (5) that

\[
A^{(n)} = 2MA^{(n-1)} = 2^n \ M^n \ A. \tag{6}
\]

where clearly

\[
A^{(0)} = \left[ \frac{1}{2}a_0 \quad a_1 \ldots \quad a_N \right]^T.
\]
The matrix equation (6) gives a relation between the Chebyshev coefficient matrix $A$ of $y(x)$ and the Chebyshev coefficient matrix $A^{(n)}$ of the $n$th derivative of $y(x)$.

We can also assume that

$$[y(x)]^m = \sum_{r=0}^{nN} a_r^n T_r(x),$$

$$[y'(x)]^m = \sum_{r=0}^{nN} a_r^{(1)m} T_r(x),$$

$$[y''(x)]^m = \sum_{r=0}^{nN} a_r^{(2)s} T_r(x),$$

The recurrence relation between the Chebyshev coefficients $a_r^n$, $a_r^{(1)m}$, $a_r^{(2)s}$ and $a_r$ of $[y(x)]^m$, $[y'(x)]^m$, $[y''(x)]^m$ and $[y(x)]$, is given by

$$a_r^n = \sum_{i=0}^{nN} a_i^n (a_{i+r}^{n-1} + a_{r-i}^{n-1}),$$

where $i + r \leq nN$

$$a_r^{(1)m} = \sum_{i=0}^{nN} a_i^{(1)} (a_{i+r}^{(1)m-1} + a_{r-i}^{(1)m-1}),$$

where $i + r \leq mN$

$$a_r^{(2)s} = \sum_{i=0}^{nN} a_i^{(2)} (a_{i+r}^{(2)s-2} + a_{r-i}^{(2)s-2}),$$

where $i + r \leq sN$

$$\sum_{k=1}^{I} \sum_{i=0}^{l} p_{ik} x^i [y''(x)]^k + \sum_{k=1}^{m} \sum_{i=0}^{l} q_{ik} x^i [y'(x)]^k + \sum_{k=1}^{n} \sum_{i=0}^{l} r_{ik} x^i [y(x)]^k = f(x)$$

and $a_{r-i}^{n-1} = 0$, $a_{r-i}^{(1)m-1} = 0$, $a_{r-i}^{(2)s-2} = 0$ for $r - i \leq 0$.

Now let us take $r = 0,1,\ldots,N$ and assume $a_r^n = 0$, $a_r^{(1)m} = 0$, $a_r^{(2)s} = 0$ and $a_r = 0$

for $r > N$. Then the system can be transformed into matrix form

$$A^n = \begin{bmatrix} \frac{1}{2} a_0^n & a_1^n & \ldots & a_N^n \end{bmatrix}^T,$$

$$A^{(1)m} = \begin{bmatrix} \frac{1}{2} a_0^{(1)m} & a_1^{(1)m} & \ldots & a_N^{(1)m} \end{bmatrix}^T,$$

$$A^{(2)s} = \begin{bmatrix} \frac{1}{2} a_0^{(2)s} & a_1^{(2)s} & \ldots & a_N^{(2)s} \end{bmatrix}^T.$$

3. METHOD OF SOLUTION

To obtain the solution of Eq. (1) in the form of expression (3) we first reduce Eq.(1) to a differential equation whose coefficients are polynomials. For this purpose, we assume that the functions $P_k(x)$, $Q_k(x)$, and $R_k(x)$ can be expressed in the forms

$$P_k(x) = \sum_{i=0}^{l} p_{ik} x^i$$

$$Q_k(x) = \sum_{i=0}^{l} q_{ik} x^i$$

$$R_k(x) = \sum_{i=0}^{l} r_{ik} x^i$$

which are Taylor polynomials of degree $I$. By using the expressions (7) in Eq. (1), we get

$$\sum_{k=1}^{I} \sum_{i=0}^{l} p_{ik} x^i [y''(x)]^k + \sum_{k=1}^{m} \sum_{i=0}^{l} q_{ik} x^i [y'(x)]^k + \sum_{k=1}^{n} \sum_{i=0}^{l} r_{ik} x^i [y(x)]^k = f(x)$$

and $M_x = [m_{ij}]$, $(i = 0,1,\ldots,N+1$ and $j = 0,1,\ldots,N+1$) is a matrix of size $(N+1) \times (N+1)$. The elements of $M_x$ are given in [3].

Also we assume that the function $f(x)$ can be expanded as

$$f(x) = \sum_{r=0}^{N} f_r T_r(x)$$

or in the matrix form

$$[f(x)] = T_1 F$$

where

$$T_1 = \begin{bmatrix} T_0(x) & T_1(x) & T_2(x) & \ldots & T_N(x) \end{bmatrix}.$$
Substituting the expressions (9) and (10) into Eq. (8) and simplifying the result, we have the matrix equation

\[
F = \left[ \frac{1}{2} f_0 \ f_1 \ \ldots \ f_N \right]^T.
\]

which corresponds to a system of \((N+1)\) algebraic equations for the unknown Chebyshev coefficients \(a_r, r = 0,1,\ldots, N\).

Briefly, we can write this equation in the form

\[
W = F
\]

where

\[
W = \begin{bmatrix} w_0 & \frac{1}{2} f_0 \\ w_1 & f_1 \\ \vdots & \vdots \\ w_N & f_N \end{bmatrix}
\]

Then the augmented matrix of Eq. (12) becomes

\[
[W; F] = \begin{bmatrix} w_0 & \frac{1}{2} f_0 \\ w_1 & f_1 \\ \vdots & \vdots \\ w_N & f_N \end{bmatrix}
\]

Next we can obtain the corresponding matrix forms for the conditions (2) as follows: The expression (3) and its derivative are equivalent to the matrix equations

\[
y^{(0)}(x) = \begin{bmatrix} T_0(x) & T_1(x) & \ldots & T_N(x) \end{bmatrix} A,
\]

\[
y^{(1)}(x) = 2 \begin{bmatrix} T_0(x) & T_1(x) & \ldots & T_N(x) \end{bmatrix} MA
\]

and

\[
y^{(2)}(x) = 4 \begin{bmatrix} T_0(x) & T_1(x) & \ldots & T_N(x) \end{bmatrix} M^2 A
\]

where

\[
A = \begin{bmatrix} \frac{1}{2} a_0 & a_1 & \ldots & a_N \end{bmatrix}
\]

By using these equations, the quantities \(y^{(i)}(a), y^{(i)}(b)\) and \(i = 1,2\) can be written as

where \(u_j\) and \(v_j\) are related to the coefficients \(a_i, b_i, \alpha_i, \beta_i\) in Eq. (2). Of course, we should be careful in the choice of coefficients of the conditions given by Eq. (2).
Consequently, by replacing the two rows matrices (10) by the last two rows of the augmented matrix (13), we have

\[
W^* = \begin{bmatrix}
w_0 & 1/2 f_0 \\
w_1 & f_1 \\
\vdots & \vdots \\
w_{N-2} & f_{N-2} \\
u_0 & \lambda \\
v_0 & \mu
\end{bmatrix}
\]

and thus by solving the system of \((N + 1)\) algebraic equations, the matrix \(A\) (thereby the coefficients \(a_r\)) is determined.

4. EXAMPLES

The method is demonstrated by following examples.

Example 4.1. Let us consider the initial value problem

\[
y'' + y^2 = x^2 + 2x + 1, \quad -1 \leq x \leq 1 \quad \text{and} \quad y(0) = 1, \quad y'(0) = 1
\]

(17)

The matrix equation for (11) becomes

\[
\{A^{(2)} - A^2\} = F.
\]

\[
\begin{pmatrix}
0 & 0 & 4 & 0 & 32 \frac{a_0}{2} \\
0 & 0 & 0 & 24 & 0 \frac{a_1}{1} \\
0 & 0 & 0 & 0 & 48 \frac{a_2}{a_2} \\
0 & 0 & 0 & 0 & 0 \frac{a_3}{a_3} \\
0 & 0 & 0 & 0 & 0 \frac{a_4}{a_4}
\end{pmatrix}
+ \begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1
\end{pmatrix}
= \begin{pmatrix}
\frac{1}{2} (\frac{1}{2} a_0^2 + a_1^2 + a_2^2 + a_3^2 + a_4^2) \\
\frac{1}{2} a_0 a_1 + a_1 a_2 + a_2 a_3 + a_3 a_4 \\
\frac{1}{2} a_0 a_3 + a_1 a_2 + a_3 a_4 \\
\frac{1}{2} a_0 a_3 + a_1 a_2 + a_3 a_4 \\
\frac{1}{2} a_0 a_3 + a_1 a_2 + a_3 a_4
\end{pmatrix}
= \begin{pmatrix}
3 \\
\frac{2}{2} \\
\frac{2}{2} \\
\frac{2}{2} \\
0
\end{pmatrix}.
\]

and approximate the solution \(y(x)\) by the Chebyshev polynomial

\[
y(x) = \sum_{r=0}^{4} a_r T_r(x)
\]

where \(N = 2\).
\[
\begin{bmatrix}
4a_2 + 32a_4 + \frac{1}{2} \left( \frac{1}{2}a_0^2 + a_1^2 + a_2^2 + a_3^2 + a_4^2 \right) & ; & \frac{3}{2} \\
24a_3 + \left( a_0a_1 + a_1a_2 + a_2a_3 + a_3a_4 \right) & ; & \frac{3}{2} \\
48a_4 + \left( \frac{1}{2}a_1^2 + a_0a_2 + a_1a_3 \right) & ; & \frac{3}{2} \\
\left( a_0a_3 + a_1a_2 + a_2a_4 \right) & ; & 0 \\
\left( \frac{1}{2}a_2^2 + a_1a_3 + a_0a_4 \right) & ; & 0 \\
\end{bmatrix}
\]

and the augmented matrices corresponding to the conditions

\begin{align*}
y(0) & = 1, \quad y'(0) = 1 \\
\end{align*}

are obtained as

\[
\bar{U} = \begin{bmatrix} \frac{1}{2}a_0 - a_2 + a_4 & ; & 1 \end{bmatrix}
\text{and } \bar{V} = \begin{bmatrix} a_1 - 3a_3 & ; & 1 \end{bmatrix}.
\]

\[
\bar{W}^* = \begin{bmatrix}
4a_2 + 32a_4 + \frac{1}{2} \left( \frac{1}{2}a_0^2 + a_1^2 + a_2^2 + a_3^2 + a_4^2 \right) & ; & \frac{3}{2} \\
24a_3 + \left( a_0a_1 + a_1a_2 + a_2a_3 + a_3a_4 \right) & ; & \frac{3}{2} \\
48a_4 + \left( \frac{1}{2}a_1^2 + a_0a_2 + a_1a_3 \right) & ; & \frac{3}{2} \\
\left( a_0a_3 + a_1a_2 + a_2a_4 \right) & ; & 0 \\
\left( \frac{1}{2}a_2^2 + a_1a_3 + a_0a_4 \right) & ; & 0 \\
\end{bmatrix}
\]

and has the solution

\begin{align*}
a_0 & = 2, \quad a_1 = 1, \quad a_2 = 0, \quad a_3 = 0. \\
\text{and } a_4 & = 0
\end{align*}

where \( N = 2 \).

The matrix equation for (11) becomes

\[
\left\{ \begin{array}{c}
4A^{(2)} - 2A^{(1)} + A \end{array} \right\} = F.
\]
It is safe to report that Chebyshev polynomial solution is congruent to exact solution of example 4.2.

5. CONCLUSIONS

In this paper, the usefulness of Chebyshev-matrix method for the polynomial solutions of second order nonlinear differential equations is discussed. These equations are usually difficult to solve analytically. In many cases, it is required to approximate solutions. A considerable advantage of the method is that Chebyshev coefficients of the solution are found very easily by using the computer programs. Following similar way, we can find the relations between Chebyshev coefficients, for the functions \( f(x) \) defined in \( 0 \leq x \leq 1 \) and \( 0 \leq x, y \leq 1 \), respectively. The method can be also extended to the polynomial solutions of second order nonlinear differential equations in general form.

REFERENCES

