On the Matrices with Harmonic Numbers

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ABSTRACT
In this study, firstly we define $n\times n$ matrices $P$ and $Q$ associated with harmonic numbers such that

$P = (p_{ij}) = [H_i]_{i,j=1}^n$ and $Q = (q_{ij}) = [H_{i+j}]_{i,j=1}^n$, where $H_k$ is denote $k$th harmonic number. After we study the spectral norms, Euclidean norms and determinants of these matrices.

Key Words: Harmonic number, Norm, Determinant.

1. INTRODUCTION
The harmonic numbers are defined by

$H_0 = 0$ and $H_n = \sum_{k=1}^{n} \frac{1}{k}$ for $n = 1, 2, \ldots$.

A generating function for the harmonic numbers is

$$-\frac{\ln(1-x)}{1-x}.$$ The first few harmonic numbers are 1, 3, 11, 25, 49, 137, 20, .... Harmonic numbers have many interesting properties [3,6]. For $n \geq 1$, some of them are the following

$\sum_{k=1}^{n} H_k = nH_n - n$ ,

$\sum_{k=1}^{n} \left( \frac{k}{m} \right) H_k = \left( \frac{n}{m+1} \right) \left( H_n - \frac{1}{m+1} \right)$ ,

$\sum_{k=1}^{n} \left( \frac{1}{k} \right) H_k = 2^n - \left( H_n - \frac{1}{2^n} \right)$ ,

$\sum_{k=1}^{n} \left( -1 \right)^k H_k = -\frac{1}{n}$ ,

$H_{n+1}^2 - H_n^2 = \left( \frac{1}{n+1} \right)^2 + \frac{2}{n+1}H_n$ .

The harmonic numbers have been generalized by many authors in recent works [1,2,3,5] such that

$H_0^{(r)} = 0$ and $H_n^{(r)} = \sum_{k=1}^{n} \frac{1}{k^r}$ for $n, r = 1, 2, \ldots$,

$H_n^{(2r)}> = \frac{1}{n}$ and $H_n^{(2r)} = \sum_{k=1}^{n} H_k^{(2r)}$ for $n, r = 1, 2, \ldots$.
For \(r = 0\) or \(r = 1\), these generalizations are reduced to the ordinary harmonic numbers. There are many connections between these generalizations, Stirling numbers and ordinary harmonic numbers [4,5].

For the generalized harmonic numbers \(H_\nu\), given by (1.1), Cheon and El-Mikkawy [4] defined an \(nxn\) matrix \(H\) as follows:

\[
H = \begin{pmatrix}
1 & 1 & \cdots & 1 \\
\frac{3}{2} & \frac{3}{2} & \cdots & \frac{3}{2} \\
\frac{11}{6} & \frac{11}{6} & \cdots & \frac{11}{6} \\
H_2 & H_3 & \cdots & H_n
\end{pmatrix}
\]

Moreover they characterized the inverse of the matrix \(H\).

In this paper, firstly we define the \(nxn\) matrices \(P\) and \(Q\) which entries are consist of harmonic numbers, such that these matrices are of the forms:

\[
P = \begin{pmatrix}
1 & 1 & \cdots & 1 \\
\frac{3}{2} & \frac{3}{2} & \cdots & \frac{3}{2} \\
\frac{11}{6} & \frac{11}{6} & \cdots & \frac{11}{6} \\
H_2 & H_3 & \cdots & H_n
\end{pmatrix}
\]

\[
Q = \begin{pmatrix}
\frac{3}{2} & \frac{11}{6} & \cdots & H_{n+1} \\
\frac{11}{6} & \frac{25}{12} & \cdots & H_{n+2} \\
\vdots & \vdots & \ddots & \vdots \\
H_{n+1} & H_{n+2} & \cdots & H_{2n}
\end{pmatrix}
\]

After we study these matrices.

Now, we start with some preliminaries. Let \(A = (a_{ij})\) be any \(mxn\) matrix. The \(\ell_p\) norms of the matrix \(A\) are defined by \(\|A\|_p = \left(\sum_{i=1}^{m} \sum_{j=1}^{n} |a_{ij}|^p \right)^{1/p}\), \((1 \leq p < \infty)\). For \(p = 2\), the \(\ell_2\) norm is called Euclidean norm. Also the spectral norm of the matrix \(A\) is \(\|A\| = \max \lambda(A'A)\), where \(A'\) is the conjugate transpose of the matrix \(A\). A function \(\psi\) is called a psi (or digamma) function if \(\psi(x) = \frac{d}{dx} \left[ \ln \Gamma(x) \right]\), where \(\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} dt\).

Throughout this paper, \(P\) and \(Q\) denote the matrices in (1.2) and (1.3) respectively.

2. MAIN RESULTS

**Theorem 2.1.** The eigenvalues of the \(nxn\) matrix \(P\) are

\[
\lambda_i = (n + 1)(H_{n+1} - 1),
\]

where \(m = 2,3,\ldots,n\).

**Proof.** The eigenvalues of the matrix \(P\) are roots of the equation \(|\lambda I - P| = 0\), such that

\[
\begin{vmatrix}
\lambda - 1 & -1 & \cdots & -1 & -1 \\
-\frac{3}{2} & \lambda - \frac{3}{2} & \cdots & -\frac{3}{2} & -\frac{3}{2} \\
-\frac{11}{6} & -\frac{11}{6} & \cdots & -\frac{11}{6} & -\frac{11}{6} \\
-H_n & -H_{n+1} & \cdots & \lambda - H_{n+1} & -H_n \\
-H_n & H_n & \cdots & -H_n & H_n
\end{vmatrix} = 0.
\]

From the properties of the determinant, we have

\[
|\lambda I - P| = \lambda^{n+1} \begin{vmatrix}
\lambda - 1 & -1 & \cdots & -1 & -1 \\
-\frac{3}{2} & \lambda - \frac{3}{2} & \cdots & -\frac{3}{2} & -\frac{3}{2} \\
-\frac{11}{6} & -\frac{11}{6} & \cdots & -\frac{11}{6} & -\frac{11}{6} \\
-H_n & -H_{n+1} & \cdots & \lambda - H_{n+1} & -H_n \\
-H_n & H_n & \cdots & -H_n & H_n
\end{vmatrix} = 0.
\]

If we calculate the last determinant, we obtain

\[
|\lambda I - P| = \lambda^{n+1} \begin{vmatrix}
\lambda - 1 & -1 & \cdots & -1 & -1 \\
-\frac{3}{2} & \lambda - \frac{3}{2} & \cdots & -\frac{3}{2} & -\frac{3}{2} \\
-\frac{11}{6} & -\frac{11}{6} & \cdots & -\frac{11}{6} & -\frac{11}{6} \\
-H_n & -H_{n+1} & \cdots & \lambda - H_{n+1} & -H_n \\
-H_n & H_n & \cdots & -H_n & H_n
\end{vmatrix} = 0.
\]

If we solve the equation

\[
|\lambda I - P| = \lambda^{n+1} \begin{vmatrix}
\lambda - 1 & -1 & \cdots & -1 & -1 \\
-\frac{3}{2} & \lambda - \frac{3}{2} & \cdots & -\frac{3}{2} & -\frac{3}{2} \\
-\frac{11}{6} & -\frac{11}{6} & \cdots & -\frac{11}{6} & -\frac{11}{6} \\
-H_n & -H_{n+1} & \cdots & \lambda - H_{n+1} & -H_n \\
-H_n & H_n & \cdots & -H_n & H_n
\end{vmatrix} = 0,
\]

the eigenvalues of the matrix \(P\) are

\[
\lambda_i = 1 + \frac{3}{2} + \frac{11}{6} + \cdots + \frac{1}{\sum_{k=1}^{n}} = 0,
\]

where \(m = 2,3,\ldots,n\).

**Corollary 2.1.** The determinant of the matrix \(P\) is zero.

**Lemma 2.1.** Let \(H_k\) be \(k\)th harmonic number, then the following equality holds
\[ \sum_{k=1}^{n} H_{k}^2 = (n+1)H_{n+1}^2 - (2n+3)H_{n+1} + 2n + 2. \]

**Proof.** We have \( H_k = \psi(k+1) + C \), where \( \psi \) is the digamma function and \( C \) is Euler’s constant. Then,

\[ \sum_{k=1}^{n} H_{k}^2 = \sum_{k=1}^{n} (\psi(k+1) + C)^2 \]

\[ = (n+1)(\psi(n+2) + C)^2 - (2n+3)(\psi(n+2) + C) + 2(n+1) \]

\[ = (n+1)H_{n+1}^2 - (2n+3)H_{n+1} + 2n + 2. \]

**Theorem 2.2.** The Euclidean norm of the matrix \( P \) is

\[ \|P\| = \sqrt{\sum_{k=1}^{n} (n+1)H_{n+1}^2 - (2n+3)H_{n+1} + 2n + 2}. \]

**Proof.** From the definition of the Euclidean norm, we have

\[ \|P\| = n^2 \sum_{k=1}^{n} H_{k}^2. \] (2.1)

From (2.1) and Lemma 2.1., we have

\[ \|P\| = \sqrt{n^2 \sum_{k=1}^{n} (n+1)H_{n+1}^2 - (2n+3)H_{n+1} + 2n + 2}. \]

Thus, the proof is completed.

**Theorem 2.3.** The singular values of the \( nxn \) matrix \( P \) satisfy the following equalities

\[ \sigma_i = \sqrt{n^2 \sum_{k=1}^{n} (n+1)H_{n+1}^2 - (2n+3)H_{n+1} + 2n + 2}, \]

\[ \sigma_m = 0, \]

where \( m = 2, 3, \ldots, n \).

**Proof.** The singular values of the matrix \( P \) are the square roots of the eigenvalues of the matrix \( P^H P \), where \( P^H \) is the conjugate transpose of the matrix \( P \). The matrix \( P^H P \) is of the form:

\[ P^H P = \begin{bmatrix} \alpha & \alpha & \cdots & \alpha & \alpha \\ \alpha & \alpha & \cdots & \alpha & \alpha \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ \alpha & \alpha & \cdots & \alpha & \alpha \\ \alpha & \alpha & \cdots & \alpha & \alpha \end{bmatrix}, \]

where \( \alpha = \sum_{k=1}^{n} H_{k}^2. \) Since \( \alpha = \sum_{k=1}^{n} H_{k}^2 \) and from Lemma 2.1., we have

\[ \alpha = (n+1)H_{n+1}^2 - (2n+3)H_{n+1} + 2n + 2. \] (2.2)

The eigenvalues of the matrix \( P^H P \) are

\[ \lambda_i = n\alpha, \]

\[ \lambda_m = 0, \] (2.3)

where \( m = 2, 3, \ldots, n \). From (2.2) and (2.3), the singular values of the matrix \( P \) are

\[ \sigma_i = \sqrt{n^2 \sum_{k=1}^{n} (n+1)H_{n+1}^2 - (2n+3)H_{n+1} + 2n + 2}, \]

\[ \sigma_m = 0 \]

where \( m = 2, 3, \ldots, n \).

**Corollary 2.2.** The spectral norm of the \( nxn \) matrix \( P \) is

\[ \|P\| = \sqrt{n^2 \sum_{k=1}^{n} (n+1)H_{n+1}^2 - (2n+3)H_{n+1} + 2n + 2}. \]

**Corollary 2.3.** The spectral norm of the \( nxn \) matrix \( P \) is equal to its Euclidean norm.

**Lemma 2.2.** Let \( H_k \) be the \( k \)th harmonic number, then the following equality holds

\[ \sum_{k=1}^{n} kH_{k}^2 = \frac{n^2 + n - 2}{2}H_{n+1}^2 - \frac{n^3 - 3n - 7}{2}H_{n+1} + \frac{n^3 - 9n - 10}{4}. \]

**Proof.**

\[ \sum_{k=1}^{n} kH_{k}^2 = \sum_{k=1}^{n} \left[ (k+2) + \gamma \right]^2 \]

\[ = \frac{n^2 + n - 2}{2}H_{n+1}^2 - \frac{n^3 - 3n - 7}{2}H_{n+1} + \frac{n^3 - 9n - 10}{4}. \]

**Lemma 2.3.** Let \( H_{k} \) be the \( k \)th harmonic number, then the equality

\[ \sum_{k=1}^{n} (n-k)H_{k}^2 = \sum_{k=1}^{n} \left[ (n-k+2)+\gamma \right]^2 \]

\[ = \frac{n^2 + 2n - 1}{2}H_{n+1}^2 - \frac{n^3 - 3n - 7}{2}H_{n+1} + \frac{n^3 - 9n - 10}{4}. \]

**Theorem 2.4.** For the Euclidean norm of the \( nxn \) matrix \( Q \)

\[ \|Q\| = \sqrt{n^2 \sum_{k=1}^{n} (n-k)H_{k}^2 - \left( \frac{n^2 + 2n - 1}{2}H_{n+1}^2 - \frac{n^3 - 3n - 7}{2}H_{n+1} + \frac{n^3 - 9n - 10}{4} \right)} \]

is valid.

**Proof.** From the definition of the Euclidean norm, we have
\[ \left\lfloor \frac{k}{n} \right\rfloor = \sum_{k=1}^{n} k H_{n+k}^2 + \sum_{k=1}^{n-1} (n-k) H_{n+k+1}^2. \] (2.4)

From (2.4), Lemma 2.2. and Lemma 2.3.,

\[ \left\lceil (2x + 3x + 1) H_{n+1} - (3x + 3 + 2) H_{n+1} - \left( 2x + 7 + \frac{3}{2} \right) H_{n+1} \right\rceil \right\lceil (3x + 9n + 7) H_{n+1} + \frac{14n^2 - 12n - 20}{4} \right\rfloor \]

is valid.

**Theorem 2.5.** The determinant of the matrix \( Q \) is

\[ \det(Q) = (-1)^{n+1} \prod_{i=1}^{n} \left( \frac{(2x + 2)(x+1)!}{(n+x+1)!} \right)^2 \left( 2H_{n+1} - \frac{1}{n(n+1)} \right). \]

**Proof.** If we apply Gauss Elimination Method to the determinant of the matrix \( Q \), we have

\[ \begin{vmatrix} \alpha_1 \beta_1 & \alpha_2 \beta_2 & \cdots & \alpha_n \beta_n \\ \alpha_2 & \cdots & \beta_1 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \alpha_n \beta_n \\ \end{vmatrix}, \]

where \( \alpha_i = 1, \ \beta_i = H_1 + H_i, \ \beta_i = H_{n+i} + H_{n+i+1}, \ \alpha_i = \frac{(k-1)![(k+1)!]^2(2k)!}{2k(k+1)(2k-1)!} \) and \( (k = 2, 3, \ldots, n) \).

Hence we write

\[ \det(Q) = (-1)^{n+1} \prod_{i=1}^{n} \alpha_i \beta_i = (-1)^{n+1} \prod_{i=1}^{n} \alpha_i \prod_{i=1}^{n} \beta_i. \]

Since

\[ \prod_{i=1}^{n} \alpha_i = \prod_{i=1}^{n} \left( \frac{(2x + 2)(x+1)!}{(n+x+1)!} \right)^2 \]

and

\[ \prod_{i=1}^{n} \beta_i = (H_{n+1} + H_{n+2}) = \left( 2H_{n+1} - \frac{1}{n(n+1)} \right), \]

we obtain

\[ \det(Q) = (-1)^{n+1} \prod_{i=1}^{n} \left( \frac{(2x + 2)(x+1)!}{(n+x+1)!} \right)^2 \left( 2H_{n+1} - \frac{1}{n(n+1)} \right). \]

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