Structural Stability for a Class of Nonlinear Wave Equations

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ABSTRACT

In this paper we discuss the structural stability of an initial value problem defined for the equation

\[ u_t - u_{txx} + 4uux = \beta u_x u_{xx} + uu_{xxx} \]  

(1.1)

where \( \alpha, \beta \) are constants, \( x \in \mathbb{R}, t \in \mathbb{R}^+ \). For the choices of \( \alpha \) and \( \beta \), (1.1) describe the nonlinear shallow water waves. Upper and lower bounds are derived for energy decay rate in every finite interval \([0,T]\) which reveals that only the lower bound of the energy decays exponentially.

Key Words: Degasperis-Procesi equation, Camassa-Holm equation, traveling wave

1. INTRODUCTION

The Degasperis-Procesi (D-P) equation

\[ u_t - u_{txx} + 4uux = \beta u_x u_{xx} + uu_{xxx} \quad x \in \mathbb{R}, t > 0 \]  

(1.4)

was proposed in [1] as one out of three integral equations within a certain family of third-order nonlinear dispersive partial differential equations; the other two being the well-known Korteweg-de Vries (KdV)

\[ u_t - 6uu_x + u_{xxx} = 0 \quad x \in \mathbb{R}, t > 0 \]  

(1.2)

and Camassa-Holm (C-H) equation [2]

\[ u_t - u_{txx} + 3uux = 2uu_x + uu_{xxx} \quad x \in \mathbb{R}, t > 0 \]  

(1.3)

which models the shallow water waves.

All weak traveling wave solutions of the D-P equations are classified by Lenells [3]. Similar classification for C-H has also been done in [4]. Degasperis, Holm and Hone [5] investigated D-P equation using the method of asymptotic integrability. This equation has a form similar to C-H shallow water wave equation. The exact integrability of the equation (1.1) investigated in [5]. The solitary wave solutions for modified forms of the equations D-P and C-H are developed by Wazwaz [6].

In this work we are interested in the structural stability of the equations D-P and C-H besides the upper and lower bounds of the energy for these equations. For the structural stability, it is fundamental that one wishes to know if a small change in a coefficient of the equation or boundary data, or small change of the equations themselves will lead to a drastic change in the solution or not. In this article we have proved that

\[ u_t - u_{txx} + 4uux = \beta u_x u_{xx} + uu_{xxx} \quad x \in \mathbb{R}, t > 0 \]  

(1.4)

is structurally stable with respect to the coefficients \( \alpha \) and \( \beta \). D-P and C-H equations are attained for the choices \( \alpha = 4, \beta = 3 \) and \( \alpha = 3, \beta = 2 \) respectively. We obtain that upper and lower bounds of the energy for the solutions of equations D-P and C-H are derived in every finite interval \([0,T]\) which shows that only the lower bound of the energy decays exponentially.

2. STRUCTURAL STABILITY

Now we consider the problem

\[ u_t - u_{txx} + 4uux = \beta u_x u_{xx} + uu_{xxx} \quad u \in C_0^{4,1}(\mathbb{R} \times \mathbb{R}^+), \]  

(2.1)

\[ 0 < t < T \quad \text{for fixed } T \]  

(2.2)

where \( \alpha, \beta > 1 \) are constants, \( C_0^{4,1}(\mathbb{R} \times \mathbb{R}^+) \) is the space of functions having compact support which have fourth order and first order derivative with respect to \( x \) and \( t \) respectively. To do this, we let \( (u, \alpha, \beta) \) be the solution of the following problem

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\begin{align}
  u_t - u_{xx} + \alpha_1 u_x = \beta_1 u_{xx} + uu_{xxx} & \quad u \in C^{4,1}_0(\mathbb{R} \times \mathbb{R}^+), \quad 0 < t < T \quad \text{for fixed } T \\
  u(x,0) = u_0(x) & \quad x \in \mathbb{R} \\
\end{align}

and \((v, \alpha_2, \beta_2)\) be the solution of

\begin{align}
  v_t - v_{xx} + \alpha_2 v_x = \beta_2 v_{xx} + vv_{xxx} & \quad v \in C^{4,1}_0(\mathbb{R} \times \mathbb{R}^+), \quad 0 < t < T \quad \text{for fixed } T \\
  v(x,0) = u_0(x) & \quad x \in \mathbb{R} \\
\end{align}

where \(\alpha_1, \beta_1, \alpha_2, \beta_2 > 1\) are constants. Now, we define the difference of these solutions by \(w = u - v\), \(\alpha = \alpha_1 - \alpha_2\), \(\beta = \beta_1 - \beta_2\) where we assume that \(\alpha_1 > \alpha_2\) and \(\beta_1 > \beta_2\). Then from (2.3)-(2.6), we find that \((w, \alpha, \beta)\) satisfies the following initial value problem

\begin{align}
  w_t - w_{xx} + a u x + \alpha_2 (w u_x + w v_x) - \beta_2 (w u_{xx} + v u_{xx}) - (w u_{xxx} + v v_{xxx}) = 0 \\
  w(x,0) = 0
\end{align}

We may state our result on structural stability for the problem defined by (2.1)-(2.2) as:

**Theorem 1:** Let \(w\) be the solution of the problem (2.7) and (2.8). Then \(w\) satisfies the estimate

\begin{align}
  \|w\|^2 + 2\|w_x\|^2 + 2\|w_{xx}\|^2 + \|w_{xxx}\|^2 \leq (\alpha K_1 + \beta K_2) \left(\frac{\gamma t^2}{\gamma - 1}\right)
\end{align}

for fixed \(T\) where \(K_1, K_2\) and \(\gamma\) are positive constants and \(\|\cdot\|\) denotes the \(L_2\) norm of functions.

**Proof.** Taking the inner product of (2.7) by \(w\) yields

\begin{align}
  (w_t, w) - (w_{xx}, w) + (a u u_x + \alpha_2 (w u_x + w v_x), w) - (\beta u_{xx}, w) - (\beta_2 (w u_{xx} + v u_{xx}), w) - (w u_{xxx} + v v_{xxx}, w) = 0
\end{align}

which gives

\begin{align}
  \frac{1}{2} \frac{d}{dt} \left(\|w\|^2 + \|w_x\|^2\right) = -\alpha \int u_x w_t dx - \alpha_2 \int (w u_x + w v_x) w_t dx + \beta \int u_{xx} w_t dx + \beta_2 \int (w u_{xx} + v u_{xx}) w_t dx + \int (w u_{xxx} + v v_{xxx}) w_t dx
\end{align}

Since \(u, v \in C^{4,1}_0(\mathbb{R} \times \mathbb{R}^+)\) then there exists a generic constant such that the functions \(U\) and \(V\) together with their derivatives are all bounded by a generic constant \(D\). For the first integral on the right hand side of (2.11) we obtain,

\begin{align}
  -\alpha \int u_x w_t dx \leq \alpha C \left(\|w\|^2 + \|w_x\|^2\right)
\end{align}

utilizing Cauchy and Hölder inequalities. For the second integral in the right hand side of (2.11) we get

\begin{align}
  -\alpha_2 \int (w u_x + w v_x) w_t dx \leq \alpha_2 C \left(\|w\|^2 + \|w_x\|^2\right)
\end{align}

In a similar way, we can compute the estimates for the other terms as

\begin{align}
  \beta \int u_{xx} w_t dx \leq \beta C \left(\|w\|^2 + \|w_x\|^2\right),
\end{align}

\begin{align}
  \beta_2 \int (w u_{xx} + v u_{xx}) w_t dx \leq \beta_2 C \left(\|w\|^2 + \|w_x\|^2 + \|w_{xx}\|^2\right),
\end{align}

\begin{align}
  \int (w u_{xxx} + v v_{xxx}) w dx \leq C \left(\|w\|^2 + \|w_{xxx}\|^2\right)
\end{align}
Substituting the estimates (2.12)-(2.16) in (2.11) we find
\[
\frac{d}{dt} \left( \|u\|_{2}^2 + \|w\|_{2}^2 \right) \leq C \left\{ \|u\|_{2}^2 + \beta \|u_{xx}\|_{2}^2 \right\} + C \left\{ \|u\|_{2}^2 + \beta \left( \|u_{xx}\|_{2}^2 + \|u_{xxx}\|_{2}^2 \right) \right\} + C \left( \alpha + \beta + \alpha_2 + \beta_2 + 1 \right) \|w\|_{2}^2 + \beta_2 \|w_{xx}\|_{2}^2 + \|w_{xxx}\|_{2}^2 \right) + C \left( \alpha + \beta + \alpha_2 + \beta_2 \right) \|w_{xx}\|_{2}^2 + \|w_{xxx}\|_{2}^2 \right) \right\}
\]
(2.17)

Similarly taking the inner product of (2.7) by \(w_{xx}\), we find
\[
\frac{d}{dt} \left( \|w_{xx}\|_{2}^2 + \|w_{xxx}\|_{2}^2 \right) \leq D \left\{ \|w_{xx}\|_{2}^2 + \beta \|w_{xxx}\|_{2}^2 \right\} + C \left( \|w_{xx}\|_{2}^2 + \alpha_2 \|w_{xx}\|_{2}^2 \right) + \beta_2 \|w_{xx}\|_{2}^2 + \|w_{xxx}\|_{2}^2 \right) + C \left( \alpha + \beta + \alpha_2 + \beta_2 \right) \|w_{xx}\|_{2}^2 + \|w_{xxx}\|_{2}^2 \right) \right\}
\]
(2.18)

Now let us differentiate equation (2.7) with respect to \(x\):
\[
w_{xx} - w_{xxxxx} + \alpha(u_{x}^2 + uu_{xx}) + \alpha_2(w_{xx}u_{xx} + w_{x}u_{xxx} + v_{x}w_{xx}) - \beta(u_{x}u_{xxx} + u_{x}^2) - \beta_2(w_{xx}u_{xx} + w_{xx}u_{xxx} + v_{x}w_{xx} + v_{x}w_{xxx}) - (w_{xx} + w_{xx}u_{xx} + w_{xxx} + v_{xxx}) = 0 .
\]
(2.19)

Taking the inner product of (2.19) by \(W_{xxx}\), we find
\[
\frac{d}{dt} \left( \|w_{xx}\|_{2}^2 + \|w_{xxx}\|_{2}^2 \right) \leq C \left\{ \|w_{xx}\|_{2}^2 + \beta \|w_{xxx}\|_{2}^2 \right\} + C \left( \|w_{xx}\|_{2}^2 + \alpha_2 \|w_{xx}\|_{2}^2 \right) + \beta_2 \|w_{xx}\|_{2}^2 + \|w_{xxx}\|_{2}^2 \right) \right\}
\]
(2.20)

Adding up the inequalities (2.17), (2.18), (2.20) we have
\[
\frac{d}{dt} \left( \|u\|_{2}^2 + \|w\|_{2}^2 + \|w_{xx}\|_{2}^2 + \|w_{xxx}\|_{2}^2 \right) \leq \alpha K_1 + \beta K_2 + \gamma \|w\|_{2}^2 + \|w_{xx}\|_{2}^2 + \|w_{xxx}\|_{2}^2 \right) \right\}
\]
(2.21)

where \(K_1 = (C + D)\|u\|_{2}^2 + C \|u_{xx}\|_{2}^2 + C \|u_{xxx}\|_{2}^2\) and \(K_2 = (C + D)\|w\|_{2}^2 + C \|w_{xx}\|_{2}^2 + C \|w_{xxx}\|_{2}^2\)

and \(\gamma = C \max \{ (\alpha + \beta + 3\alpha_2 + 3\beta_2), (3\alpha + 2\beta_2 + 2\beta_2), (\alpha + \beta + \alpha_2 + 2\beta_2), (\alpha + \beta + \alpha_2 + \beta_2) \} \}

Thus, from (2.20) we have
\[
\frac{d}{dt} \Psi(t) - \gamma \Psi(t) \leq \alpha K_1 + \beta K_2
\]
(2.22)

where \(\Psi(t) = \|w\|_{2}^2 + \|w_{xx}\|_{2}^2 + \|w_{xxx}\|_{2}^2\). Solving the differential inequality (2.22), we arrive at
\[
\|w\|_{2}^2 + \|w_{xx}\|_{2}^2 + \|w_{xxx}\|_{2}^2 \leq (\alpha K_1 + \beta K_2) \left( \frac{e^{\gamma t} - 1}{\gamma} \right)
\]
for fixed \(T\). And so we have completed the proof of the theorem.

Remark 2. \(W\) and its \(X\) derivatives of order up to \(3\), tends to zero as \(\alpha \to 0, \beta \to 0\) for finite \(T\).

So, the solutions of (2.22) depend continuously on \(\alpha\) and \(\beta\) in \(C^{4,1}_0(\mathbb{R} \times \mathbb{R}^+)\) which means that (2.1)-(2.2) is structurally stable with respect to the coefficients \(\alpha\) and \(\beta\).

3. UPPER AND LOWER BOUNDS ON THE ENERGY

Let \(u\) be a solution to the initial-value problem (2.1), (2.2) with \(\alpha > 1\) and \(\beta > 1\). By similar computations given in Section 2, we find
\[
\frac{d}{dt} \left( \|u\|_{2}^2 + 2\|u_{x}\|_{2}^2 + 2\|u_{xx}\|_{2}^2 + \|u_{xxx}\|_{2}^2 \right) = (2 - \beta - \alpha)\int u^2 u_{xxx} dx - (2\beta + 1)\int u u_{xxx} dx + (1 - 2\beta - 5\alpha)\int u_x^2 dx
\]
(3.1)
If we use Cauchy and Hölder inequalities in (3.1) we obtain
\[
\begin{align*}
\frac{d}{dt} \left[ \|u\|^2 + 2\|u_x\|^2 + 2\|u_{xxx}\|^2 + \|u_{xxxx}\|^2 \right] & \geq \left(1 - \frac{\beta}{2} - \frac{\alpha}{2}\right) \max_{t \in [0,T]} |u|^2 + 2\left(\frac{5\alpha}{4} - \beta\right) \max_{t \in [0,T]} \|u_x\|^2 + 2\left(\frac{1}{4} - \frac{\beta}{2} - \frac{5\alpha}{4}\right) \max_{t \in [0,T]} \|u_{xx}\|^2 \\
& \quad + \left(1 - \frac{\beta}{2} - \frac{\alpha}{2} - \left(\frac{\alpha + 1}{2}\right) \max_{t \in [0,T]} u_{xx}^2 \right) \|u_{xxxx}\|^2 \\
\end{align*}
\]
(3.2)

Taking
\[
\eta = \max \left\{ \left(1 - \frac{\beta}{2} - \frac{\alpha}{2}\right) \max_{t \in [0,T]} |u|^2, \left(\frac{1}{4} - \frac{\beta}{2} - \frac{5\alpha}{4}\right) \max_{t \in [0,T]} \|u_x\|^2, \left(1 - \frac{\beta}{2} - \frac{\alpha}{2} - \left(\frac{\alpha + 1}{2}\right) \max_{t \in [0,T]} u_{xx}^2 \right) \right\}
\]
and
\[
Y(t) = \|u(t)\|^2 + 2\|u_x(t)\|^2 + 2\|u_{xxx}(t)\|^2 + \|u_{xxxx}(t)\|^2,
\]
we have
\[
\frac{d}{dt} Y(t) - \eta Y(t) \geq 0
\]
(3.3)

Solving the inequality (3.3) we arrive at
\[
e^{-\eta T} \left[ \|u(x,0)\|^2 + 2\|u_x(x,0)\|^2 + 2\|u_{xxx}(x,0)\|^2 + \|u_{xxxx}(x,0)\|^2 \right] \leq \|u(x,T)\|^2 + 2\|u_x(x,T)\|^2 + 2\|u_{xxx}(x,T)\|^2 + \|u_{xxxx}(x,T)\|^2
\]
(3.4)

where \(\eta \leq 0\). This inequality gives a lower bound for the energy.

Now we will derive an upper bound for the energy. From (3.1) we have
\[
\begin{align*}
\frac{d}{dt} \left[ \|u\|^2 + 2\|u_x\|^2 + 2\|u_{xxx}\|^2 \right] & \leq \left|1 - \frac{\beta + \alpha}{2}\right| \max_{t \in [0,T]} |u|^2 + 2\left\{\left[1 - \frac{5\alpha - 2\beta}{4}\right] \max_{t \in [0,T]} \|u_x\|^2 + \left(\frac{\beta}{2} + \frac{1}{4}\right) \max_{t \in [0,T]} \|u_{xx}\|^2 \right\} \|u_{xx}^2\|^2 \\
& \quad + 2\left[1 - \frac{5\alpha - 2\beta}{4}\right] \max_{t \in [0,T]} \|u_{xx}\|^2 \|u_{xxx}\|^2 + \left[1 - \frac{\beta + \alpha}{2}\right] \max_{t \in [0,T]} \|u_{xxx}\|^2 \|u_{xxxx}\|^2 \\
\end{align*}
\]
(3.5)

Taking
\[
\mu = \max \left\{ \left|1 - \frac{\beta + \alpha}{2}\right| \max_{t \in [0,T]} |u|^2, \frac{1 - 5\alpha - 2\beta}{4} \max_{t \in [0,T]} \|u_x\|^2 + \left(\frac{\beta}{2} + \frac{1}{4}\right) \max_{t \in [0,T]} \|u_{xx}\|^2, \frac{1 - 5\alpha - 2\beta}{4} \max_{t \in [0,T]} \|u_{xx}\|^2 \right\}
\]
\[
\left[1 - \frac{\beta + \alpha}{2}\right] \max_{t \in [0,T]} \|u_{xx}\|^2 + \left(\frac{\beta}{2} + \frac{1}{4}\right) \max_{t \in [0,T]} \|u_{xxx}\|^2
\]
and
\[
Y(t) = \|u(t)\|^2 + 2\|u_x(t)\|^2 + 2\|u_{xxx}(t)\|^2 + \|u_{xxxx}(t)\|^2,
\]
we have
\[
\frac{d}{dt} Y(t) - \mu Y(t) \leq 0
\]
(3.6)

Then integrating the inequality (3.6) from 0 to \(T\) we arrive at
\[
\|u(x,T)\|^2 + 2\|u_x(x,T)\|^2 + 2\|u_{xxx}(x,T)\|^2 + \|u_{xxxx}(x,T)\|^2 \leq e^{\eta T} \left[ \|u(x,0)\|^2 + 2\|u_x(x,0)\|^2 + 2\|u_{xxx}(x,0)\|^2 + \|u_{xxxx}(x,0)\|^2 \right]
\]
(3.7)

where \(\mu \geq 0\). (3.7) gives an upper bound for the energy in every finite interval \([0,T]\).

We may combine the above results as in the following theorem.
Theorem 3. The energy corresponding to the solutions of the initial value problem (2.1)-(2.2) in $C^{4,1}_0(\mathbb{R} \times \mathbb{R}^+)$ satisfy
\[
e^{2T} \left[ \|u(x,0)\|^2 + 2\|u_x(x,0)\|^2 + 2\|u_{xx}(x,0)\|^2 + \|u_{xxx}(x,0)\|^2 \right] \leq \|u(x,T)\|^2 + 2\|u_x(x,T)\|^2 + 2\|u_{xx}(x,T)\|^2 + \|u_{xxx}(x,T)\|^2 \leq e^{2T} \left[ \|u(x,0)\|^2 + 2\|u_x(x,0)\|^2 + 2\|u_{xx}(x,0)\|^2 + \|u_{xxx}(x,0)\|^2 \right]
\]
For fixed $T$ where $\alpha, \beta > 1$ are constants.

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