The Riesz Core of a Sequence

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ABSTRACT

The Riesz sequence space $\ell^q_r$ including the space $c$ has recently been defined in [14] and its some properties have been investigated. In the present paper, we introduce a new type core, $K_q$-core, of a complex valued sequence and also determine the required conditions for a matrix $B$ for which $K_q$-core $(Bx) \subseteq K_q$-core $(x)$, and $K_q$-core $(Bx) \subseteq stA$-core $(x)$ and $K_q$-core $(Bx) \subseteq K_q$-core $(x)$ hold for all $x \in \ell_\infty$.

Keywords: Matrix transformations, core of a sequence, statistical convergence

1. INTRODUCTION

Let $E$ be a subset of $N=\{0,1,2,...\}$. Natural density $\delta$ of $E$ is defined by

$$
\delta(E) = \lim_{n \to \infty} \frac{1}{n} \left| \{k \leq n : k \in E\} \right|
$$

where the vertical bars indicate the number of elements in the enclosed set. A sequence $x = (x_k)$ is said to be statistically convergent to the number $A$ if for every $\varepsilon > 0$ the set $\delta \{k : |x_k - A| \geq \varepsilon\}$ has $A$-density zero, [4]. In this case, we write $stA-lim x = s$. By $st(A)$ and $st(A)_0$ we respectively denote the sets of all $A$-statistically convergent and $A$-statistically null sequences.

In [8], the notion of the statistical core of a complex valued sequence introduced by Fridy and Orhan [11] has been extended to the $A$-statistical core (or $stA$-core) and it is shown for a $A$-statistically bounded sequence $x$ that

$$
stA-core(x) = \bigcap_{z \in \mathbb{C}} C_x(z),
$$

where $C_x(z) = \{w : |z - w| \leq \limsup |x_k - z|\}$. The inequalities related to the core of a sequence have been studied by many authors. For instance, see [1, 5, 6, 7].

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Let $x=(x_k)$ be a sequence in $C$, the set of all complex numbers, and $R_\delta$ be the least convex closed region of complex plane containing $x_0$, $x_1$, $x_2$,.... The Knopp Core (or $K$-core) of $x$ is defined by the intersection of all $R_k (k=1,2,...)$, [3, p.137]. In [15], it is shown that

$$
K-core(x) = \bigcap_{z \in \mathbb{C}} B_x(z)
$$

for any bounded sequence $x=(x_k)$, where $B_x(z) = \{w : |w-z| \leq \limsup |x_k - z|\}$.
7, 8, 11, 15] and the others. The matrix \( R = (r_{nk}) \) defined by
\[
r_{nk} = \begin{cases} \frac{q_k}{Q_n}, & k \leq n \\ 0, & k > n \end{cases}
\]
is called Riesz matrix and denoted by \((R, q_k)\) or shortly \(R\), where \(q_k\) is a sequence of non-negative numbers which are not all zero and \(Q_n = q_1 + q_2 + \cdots + q_n\), \(n \in N\), \(q_k > 0\). It is well-known that \(R\) is regular if and only if \(\lim_{n} Q_n = \infty\) [14].

Using the convergence domain of the Riesz matrix, the new sequence spaces \(c_0^{n}\) and \(c_0^{n}\) respectively including the spaces \(c\) and \(c_0\) have been constructed by Malkowsky & Raković in [13] and Altay & Başar in [2] and their some properties have been investigated, where \(c\) and \(c_0\) are the spaces of all convergent and null sequences, respectively.

Let \(B\) be an infinite matrix of complex entries \(b_{nk}\) and \(x = (x_k)\) be a sequence of complex numbers. Then \((Bx)_n = \sum_{k} b_{nk} x_k\) converges for each \(x \in R\). For two sequence spaces \(X\) and \(Y\) we say that \(B = (b_{nk}) \in (X, Y)\) if \(Bx \in Y\) for each \(x \in X\). If \(X\) and \(Y\) are equipped with the limits \(X\)-lim and \(Y\)-lim respectively, \(B = (b_{nk}) \in (X, Y)\) and \(Y\)-lim, \(Bx_\alpha = X\)-lim, \(sk\) for each \(x = (x_k) \in X\), then we say \(B\) regularly transforms \(X\) into \(Y\) and write \(B = (b_{nk}) \in (X, Y\text{-lim})\).

In the present paper, we firstly introduce a new type core, \(K_{c_0}^{\text{core}}\), of a complex valued sequence and also determine the necessary and sufficient conditions on a matrix \(B\) for which \(K_{c_0}^{\text{core}}(Bx) \subseteq K_{c_0}(x)\), \(K_{c_0}^{\text{core}}(Bx) \subseteq \text{st}_{c_0}(x)\) and \(K_{c_0}^{\text{core}}(Bx) \subseteq K_{c_0}(x)\) for all \(x \in \ell_\infty\), where \(\ell_\infty\) is the space of all bounded complex sequences. To do these, we need to characterize the classes \((c, r_{\ell_\infty}^{\text{reg}}), (r_{\ell_\infty}^{\text{reg}}), (r_{\ell_\infty}^{\text{reg}})\) and \((\text{st}(A) \cap \ell_\infty, r_{\ell_\infty})\).

2. LEMMATA

In this section, we prove some lemmas which will be useful to our main results. For brevity, in what follows we write \(\bar{b}_{nk}\) in place of
\[
\frac{1}{Q_n} \sum_{k=0}^{n} q_k b_{nk} ; (n, k \in N).
\]

**Lemma 2.1.** \(B \in (\ell_\infty, r_{\ell_\infty}^{\text{reg}})\) if and only if
\[
\|B\|_p = \sup_{n} \sum_{k} |\bar{b}_{nk}| < \infty, \quad (2.1)
\]
\[
\lim_{n} \bar{b}_{nk} = a_k \quad \text{for each } k, \quad (2.2)
\]
\[
\lim_{n} \sum_{k} |\bar{b}_{nk} - a_k| = 0. \quad (2.3)
\]

**Proof.** Let \(x \in \ell_\infty\) and consider the equality
\[
\frac{1}{Q_n} \sum_{k=0}^{n} q_k \sum_{j=0}^{m} b_{kj} x_k = \frac{1}{Q_n} \sum_{j=0}^{m} q_j b_{kj} x_j ; (m, n \in N)\]
which yields as \(m \to \infty\) that
\[
\frac{1}{Q_n} \sum_{k=0}^{n} q_k (Bx)_k = (Dx)_\infty; (n \in N), \quad (2.4)
\]
where \(D = (d_{nk})\) defined by
\[
d_{nk} = \begin{cases} \frac{1}{Q_n} \sum_{j=0}^{m} q_j b_{kj}, & 0 \leq k \leq n \\ 0, & k > n. \end{cases}
\]

Therefore, one can easily see that \(B \in (\ell_\infty, r_{\ell_\infty}^{\text{reg}})\) if and only if \(D \in (\ell_\infty, c)\) (see [13]) and this completes the proof.

**Lemma 2.2.** \(B \in (c, r_{\ell_\infty}^{\text{reg}})\) if and only if the conditions (2.1) and (2.2) of the Lemma 2.1 hold with \(a_k = 0\) for all \(k \in N\) and
\[
\lim_{n} \sum_{k} \bar{b}_{nk} = 1. \quad (2.5)
\]

Since the proof is easy we omit it.

**Lemma 2.3.** \(B \in \text{(st}(A) \cap \ell_\infty, r_{\ell_\infty})\text{reg)}\) if and only if \(B \in (c, r_{\ell_\infty}^{\text{reg}})\) and
\[
\lim_{n} \sum_{k \in E} |\bar{b}_{nk}| = 0 \quad (2.6)
\]
for every \(E \subseteq N\) with \(\delta_{\infty}(E) = 0\).

**Proof (Necessity).** Because of \(c \subseteq \text{st}(A) \cap \ell_\infty, B \in (c, r_{\ell_\infty}^{\text{reg}})\). Now, for any \(x \in \ell_\infty\) and a set \(E \subseteq N\) with \(\delta_{\infty}(E) = 0\), let us define the sequence \(z = (z_k)\) by
\[
z_k = \begin{cases} x_k, & k \in E \\ 0, & k \not\in E. \end{cases}
\]
Then, since \(z \in \text{st}(A)_{\infty}, A \subseteq \ell_\infty\), \(D_{\infty}\) is in the space of sequences consisting the Riesz transforms of them in \(c_0\). Also, since
\[
\sum_{k \in E} \bar{b}_{nk} z_k = \sum_{k \in E} \bar{b}_{nk} x_k,
\]
the matrix \(D = (d_{nk})\) defined by \(d_{nk} = \bar{b}_{nk} (k \in E), 0 \neq (k \not\in E)\) in the class \((\ell_\infty, r_{\ell_\infty})\). Hence, the necessity of (2.6) follows from Lemma 2.1.

**(Sufficiency).** Let \(x \in \text{st}(A) \cap \ell_\infty\) with \(\delta_{\infty}(x) = \ell\). Then, the set \(E\) defined by \(E = \{k; |x_k| \geq \varepsilon\}\) has \(A\)-density zero and \(3|x_k| < \varepsilon\) if \(k \not\in E\). Now, we can write
\[
\sum b_{nk}x_k = \sum b_{nk}(x_k - l) + k\sum b_{nk}. \tag{2.7}
\]

Since \[|\sum b_{nk}(x_k - l)| \leq \|x\|\sum b_{nk} + \varepsilon \|B\|,\]

letting \(n \to \infty\) in (2.7) with (2.6), we have
\[
\lim_{n} \sum b_{nk}x_k = l.
\]

This implies that \(B \in (\text{st}(c) \cap \ell_{\infty}, r_{c}^{s})_{\text{reg}}\) and the proof is completed. When \(B\) is chosen as the Cesàro matrix in Lemma 2.3, we have the following corollary.

**Corollary 2.4.** \(B \in (\text{st} \cap \ell_{\infty}, r_{c}^{s})_{\text{reg}}\) if and only if \(B \in (c, r_{c}^{s})_{\text{reg}}\) and
\[
\lim_{n} \sum_{k \in E} |b_{nk}| = 0
\]
for every \(E \subset N\) with \(\delta(E) = 0\).

**Lemma 2.5.** \(B \in (r_{c}^{s}, r_{c}^{s})_{\text{reg}}\) if and only if
\[
(b_{nk}) \in cs
\]
holds and \((c) \subset (c, r_{c}^{s})\), where \(C = (c_{ak})\) is defined by
\[
c_{ak} = \Delta\left(\frac{b_{nk}}{q_{k}}\right)Q_{k}
\]
for all \(n,k \in N\) and \(cs\) is the space of all convergent series.

**Proof.** (Sufficiency). Take \(x \in r_{c}^{s}\). Then, the sequence \((b_{nk})_{k \in N} \in \ell_{\infty} \cap \ell_{C}\) for all \(n \in N\) and this implies the existence of the \(B\)-transform of \(x\).

Let us now consider the following derived equality by using the relation
\[
y_k = \sum_{i=0}^{k} \frac{q_{i}}{Q_{k}}x_{i}
\]
from the \(m^{th}\) partial sum of the series \(\sum b_{nk}x_k\),
\[
\sum_{k=0}^{m} b_{nk}x_k = \sum_{k=0}^{m-1} \Delta \left(\frac{b_{nk}}{q_{k}}\right)Q_{k}y_{k} + b_{nm}\sum_{m=0}^{m} y_{m} (m,n) \in N\),
\]
(2.9)

Then, using (2.1), we obtain from (2.9) as \(m \to \infty\) that
\[
\sum b_{nk}x_k = \sum_{k=0}^{\infty} \Delta \left(\frac{b_{nk}}{q_{k}}\right)Q_{k}y_{k},
\]
i.e. \(Bx = Cy\). Since \(x \in r^{s}\) if and only if \(y \in c\), (2.2) implies that \(B \in (r_{c}^{s}, r_{c}^{s})\).

**(Necessity).** Conversely, let \(B \in (r_{c}^{s}, r_{c}^{s})\). Then, since \((b_{nk})_{k \in N} \in \ell_{C}\) for all \(n \in N\), the necessity of (2.1) is immediate. On the other hand, (2.2) follows from (2.4).

### 3. \(K_{c}\)-Core

Let us write
\[
t_{c}^{s}(x) = A^{c}(x) = \frac{1}{Q_{\infty}} \sum_{k=0}^{\infty} q_{k}x_{k}.
\]

Then, we can define \(K_{c}\)-core of a complex sequence as follows.

**Definition 3.1.** Let \(H\) be the least closed convex hull containing \(t_{c}^{s}, \ t_{n+1}^{s}, \ t_{n+2}^{s}, \ldots\). Then, \(K_{c}\)-core of \(x\) is the intersection of all \(H_{n}\) i.e.,
\[
K_{c}\text{-core}(x) = \bigcap_{n=1}^{\infty} H_{n}.
\]

Note that, actually, we define \(K_{c}\)-core of \(x\) by the \(K\)-core of the sequence \((t_{n}^{s})\). Hence, we can construct the following theorem which is an analogue of \(K\)-core, (see [16]).

**Theorem 3.2.** For any \(z \in C\), let
\[
G_{c}(z) = \{w \in C \mid |w - z| \leq \limsup_{n} |t_{n}^{s} - z|\}.
\]

Then, for any \(x \in \ell_{\infty}\),
\[
K_{c}\text{-core}(x) = \bigcap_{z \in C} G_{c}(z).
\]

Note that in the case \(q_{z} = 1\) for all \(n\), the Riesz core is reduced to the Cesàro core.

Now, we may give some inclusion theorems.

**Theorem 3.3.** Let \(B \in (c, r_{c}^{s})_{\text{reg}}\). Then, \(K_{c}\)-core \((Bx)\) \(\subseteq K\)-core \((x)\) for all \(x \in \ell_{\infty}\) and only if
\[
\lim_{n} \sum_{k} |b_{nk}| = 1.
\]

**Proof (Necessity).** Let us define a sequence \(x = x^{(0)} = \{x_{n}^{(0)}\} \) by
\[
x_{n}^{(0)} = sgn b_{nk}x_{k}
\]
for all \(n \in N\). Then, since \(\limsup x^{(0)} = 1\) for all \(n \in N\), \(K\)-core \((x)\) \(\subseteq B(0)\). Therefore, by hypothesis,
\[
\left\{w \in C \mid |w| \leq \limsup_{n} \sum_{k} |b_{nk}| \right\} \subseteq B(0)
\]
which gives the necessity of (3.1).

**(Sufficiency).** Let \(w \in K_{c}\)-core \((Bx)\). Then, for any given \(z \in C\), we can write
\[
|w - z| \leq \limsup_{n} |t_{n}^{s} - (Bx) - z|
\]
\[
= \limsup_{n} |z - \sum_{k} b_{nk}x_{k}| \leq \limsup_{n} \sum_{k} |b_{nk}(z - x_{k})| + \limsup_{n} |z||1 - \sum_{k} b_{nk}|
\]
Thus, applying the operator $\limsup_{n}$ under the light of the hypothesis and combining (3.2) with (3.3), we have

$$\sum_{k\in E} |\tilde{b}_{nk} (x - x_k) + |\sum_{k\in k\in E} b_{nk} (z - x_k) | \leq \sup_{k\in k\in E} |\tilde{b}_{nk} | + \sum_{k\in k\in E} |b_{nk} |, \quad (3.3)$$

which means that $w \in K-core(x)$. This completes the proof.

**Theorem 3.4.** Let $B \in (st,\alpha) \cap \ell_{\infty}, r_{\alpha}^{\infty}$, then, $K_{r}\text{-}\text{core} (Bx) \subseteq st_{r}\text{-}\text{core} (x)$ for all $x \in \ell_{\infty}$ if and only if (3.1) holds.

**Proof (Necessity).** Since $st_{r}\text{-}\text{core} (x) \subseteq K_{r}\text{-}\text{core} (x)$, the necessity of the condition (3.1) follows from Theorem 3.3.

**Sufficiency.** Take $w \in K_{r}\text{-}\text{core} (Bx)$. Then, we can write again (3.2). Now, if $(w-z) = \limsup_{n} |\sum_{k\in k\in E} b_{nk} (z - x_k) | \leq \sup_{k\in k\in E} |\tilde{b}_{nk} | + \sum_{k\in k\in E} |b_{nk} |, \quad (3.4)$

Finally, combining (3.2) with (3.4), we have

$|w-z| \leq \text{(3.2)}$. Thus, $\limsup_{n} |\sum_{k\in k\in E} b_{nk} (z - x_k) | \leq \sup_{k\in k\in E} |\tilde{b}_{nk} | + \sum_{k\in k\in E} |b_{nk} |, \quad (3.3)$

\begin{align*}
\text{Theorem 3.5.} \text{ Let } B \in (r_{\infty}, r_{\infty}^{\infty})_{\text{reg}}. \text{ Then, } K_{r}\text{-}\text{core} (Bx) \\
\subseteq K_{r}\text{-}\text{core} (x) \text{ for all } x \in \ell_{\infty}, \text{ if and only if (3.1) holds.}
\end{align*}

**Proof (Necessity).** Since $K_{r}\text{-}\text{core} (x) \subseteq K_{r}\text{-}\text{core} (x)$, the necessity of the condition (3.1) follows from Theorem 3.3.

**Sufficiency.** Let $w \in K_{r}\text{-}\text{core} (Bx)$. Then, we can write (3.2). Now, if $\limsup_{n} |\sum_{k\in k\in E} b_{nk} (z - x_k) | \leq \sup_{k\in k\in E} |\tilde{b}_{nk} | + \sum_{k\in k\in E} |b_{nk} |, \quad (3.3)$

Therefore, considering the operator $\limsup_{n}$ in (3.5) and using the hypothesis, we get that $|w-z| \leq v + \varepsilon$. This means that $w \in K_{r}\text{-}\text{core} (x)$ and the proof is completed.

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**REFERENCES**


